

From the Same Points, One Method Diverges by a Thousand and the Other Converges to a Thousandth

— The Two Cures for the Runge Phenomenon Are Independent of Each Other — At $n = 29$, **1131.42** against **0.00104**, a Ratio of 10^6 — [Paper 329]

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September 2026

— *From thinking to syncing.* —

Abstract

Take one function, sample it at the same points, and add points. This paper asks what decides whether the approximation improves or worsens—the answer is that there are two things one can move, independently. No new mathematical theorem and no new law is claimed. First, from the same 29 points the polynomial gives **1131.420474** and the cubic spline **0.001039** (Section 2). Second, this is the core of the paper. There are two cures, and either one alone suffices (Section 3). Third, moving the nodes gives **0.00365254** and moving the family gives **0.00103905**. Improvements of 309762.9 and 1088894.0 (Section 3). Fourth, what breaks lies in the placement of the nodes, not in the function. The Lebesgue constant at $n = 29$ is 1803783.3902 against 3.0829 (Section 4). Fifth, the equispaced polynomial passes an error of 1 at $n = 9$. That is larger than the maximum of the function itself (Section 4). Sixth, the separator is whether the nodes were moved or the family was (Section 5).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—Runge’s example of 1901, Chebyshev nodes, cubic splines, and the Lebesgue constant are all standard. We do not build approximation theory—all we use is one function and three approximations of it. We prove no convergence theorem—the general question of which family converges under which hypotheses is not entered. Only numbers actually measured for one function are put down. We do not seek optimal nodes—that Chebyshev nodes are near-optimal is used, but no optimality is proved. We do not treat finite elements—using splines to solve differential equations is not entered. It is mentioned and no more. Rounding error is not the subject—the numbers here come out to the same digits in extended precision. Loss of significance belongs to Paper 221 and Paper 330. Relation to earlier papers: Paper 221 showed that changing the last bit moves an eigenvalue by 0.16—there the input is perturbed; here the input is exact and the approximating family collapses anyway. Both are “the numbers cannot be trusted”, from different causes. Paper 300 showed whether two things share a root is decidable—by that criterion the two cures here have distinct roots. Moving one leaves the other where it was. Paper 182 showed “the curse of dimensionality” is not one curse—the same form, with “the cure for the Runge phenomenon” not being one cure here. Paper 324 showed that no search is universal because of how the problems are counted—that every approximating family has functions it handles badly comes from the same root. Paper 316 showed the same integer returning from three different counts—there different routes reach the same answer; here different routes reach answers equally good. What is added is putting all three on one function under one way of measuring, placing the Lebesgue constant in the same table so that the “why” is a number, identifying the n at which the error passes the maximum of the function, and putting the separator on nodes against family.

1. Notation and the Question

Setting: On the interval $[-1, 1]$ take

$$f(x) = \frac{1}{1 + 25x^2}.$$

The function is smooth on $[-1, 1]$ and its values run only from $1/26$ to 1 —as a target for approximation it is not a particularly nasty shape.

Observation 1.1

More points ought to mean a better fit, one naively thinks. There is no intuitive reason why more samples should hurt—and yet the interpolating polynomial on equispaced points gets worse the more points one adds.

Observation 1.2

The way of measuring is fixed first. Error means the maximum absolute error over $[-1, 1]$ sampled at 20001 points—all three are measured with the same ruler.

Question: What is it that breaks, and what must be moved to repair it?

2. Same Points, Same Function, Two Outcomes

Calculation 2.1: Take n equispaced points and compare (a) the polynomial through them with (b) the cubic spline through them (computed in this paper).

Points n	Equispaced polynomial	Equispaced cubic spline
5	0.438357	0.317082
13	3.663393	0.006909
21	59.822309	0.003183
29	1131.420474	0.001039

Observation 2.2

The left column rises and the right one falls. The same points, the same function—and opposite directions.

Observation 2.3

At $n = 5$ there is almost nothing between them. 0.438357 against 0.317082, a factor of 1.38—tried at a small number of points, the difference is invisible.

Observation 2.4

At $n = 29$ they are apart by a factor of 1088894. 1131.420474 against 0.001039—from one and the same sample, answers six orders of magnitude apart.

Observation 2.5

What the left column means, said exactly. The values of f never exceed 1. An error of 1131 therefore means —that the approximation swings to a thousand times the height of what it approximates.

3. The Core: There Are Two Cures and They Are Independent

Setting: To repair what is broken there are two things one may move.

What is moved	What it changes
Placement of the nodes	where the samples are taken
The approximating family	which set of functions is fitted

Calculation 3.1: Fix $n = 29$ and move one at a time (computed in this paper).

	Nodes	Family	Maximum error
Baseline	equispaced	polynomial	1131.420474
Move the nodes only	Chebyshev	polynomial	0.00365254
Move the family only	equispaced	cubic spline	0.00103905

Observation 3.2

This is the core of the paper. Moving either one alone repairs it—leave the nodes equispaced and the spline gives 0.00103905; leave the family polynomial and Chebyshev nodes give 0.00365254.

Observation 3.3

The improvements are 309762.9 and 1088894.0. Both are five to six orders of magnitude—this is not a difference between working and not working; both work decisively.

Observation 3.4

The two answers are not the same. Chebyshev over spline is 3.5152—equally good, but not equal. Two separate mechanisms happen to land in neighbouring decades.

Observation 3.5

Apply the criterion of Paper 300.

Criterion	Result
Is there a correspondence preserving an identity	No. One is about nodes, one about a function space
Does the same quantity appear	No. A Lebesgue constant against a piecewise degree
Does moving one move the other	No. Changing nodes barely moves the spline error

Observation 3.6

All three rows point to distinct roots. One phrase, “the cure for the Runge phenomenon”, names two things that do not meet—the same shape Paper 182 found in “the curse of dimensionality”.

4. Why: What Breaks Does Not Lie in the Function

Setting: Does interpolation error come from a bad function or from a bad placement of points? These can be separated. There is a quantity fixed by the nodes alone, the Lebesgue constant Λ_n —an amplification factor for error computable from the positions of the points without ever looking at the function.

Calculation 4.1: Compute Λ_n for the same node sets (computed in this paper).

Points n	Equispaced	Chebyshev
5	2.2078	1.7988
13	89.3249	2.5393
21	10986.6965	2.8678
29	1803783.3902	3.0829

Observation 4.2

f does not appear once in this table. Only the positions of the points went into it——so the cause of the collapse does not lie on the side of the function being approximated.

Observation 4.3

Equispaced grows by a factor of 816995.9 from $n = 5$ to $n = 29$, while Chebyshev goes only from 1.7988 to 3.0829. Same interval, same count——placement alone moves the amplification by six orders of magnitude.

Observation 4.4

The equispaced polynomial passes an error of 1 at $n = 9$. And 1 is the maximum of f itself——past nine points, answering 0 and doing nothing is the better policy.

Observation 4.5

To be stated honestly——the spline wins here for this function. Cubic spline convergence saturates at a fixed order, so for an analytic function wanted to high accuracy Chebyshev is the faster of the two——what this paper claims is that two cures act independently, not that splines are always better.

Observation 4.6

Here is the parting from Paper 221. There, moving the last bit of the input moved the answer—here the input is exact and extended precision returns the same numbers. What is broken is not the arithmetic but the construction of the approximation.

5. The Separator: Were the Nodes Moved, or the Family?

Observation 5.1

The separator—whether the remedy changed where the samples are taken, or which set of functions is fitted.

	Changed	Freedom in nodes	How error is held down
Chebyshev nodes	position	given up	keeps Λ_n small
Cubic spline	family	kept	has no high-order terms

Observation 5.2

The third column is what divides practice. If you choose your own sample points Chebyshev is available; if you do not, only the spline remains—against data already taken, one of the two hands is missing from the start.

Observation 5.3

Both may be moved at once. And doing both merely doubles what one of them already achieved in five orders of magnitude—they do not add, because the two stop the same illness from opposite sides.

Observation 5.4

To be stated honestly——“equispaced is bad” is too strong. Equispaced is harmless while n is small (0.438357 at $n = 5$)——what is bad is not the placement but the placement together with the degree.

Observation 5.5

Where this stands relative to Paper 324. Paper 324 wrote that no search is universal because of how the problems are counted——this is the approximation-theoretic version, and that every family has functions it handles badly comes from the same root. What is shown here is that there are two independent routes around the bad cases.

Observation 5.6

One last thing——taking more samples is not the cause of this. More samples are welcome. The cause is having demanded that one single polynomial pass through all of them——the data were not bad; the demand was too strong.

6. The Census and the Close

1. 1131.420474 for the equispaced polynomial at $n = 29$ (Calculation 2.1): the function's maximum is 1.
2. 0.001039 for the spline on the same points (Calculation 2.1): the opposite direction.
3. Only a factor of 1.38 at $n = 5$ (Observation 2.3): invisible at few points.
4. 0.00365254 from moving the nodes (Calculation 3.1): family still polynomial.
5. 0.00103905 from moving the family (Calculation 3.1): nodes still equispaced.
6. Chebyshev over spline is 3.5152 (Observation 3.4): equally good, not equal.
7. Lebesgue constants 1803783.3902 against 3.0829 (Calculation 4.1): f never appears.
8. The error passes 1 at $n = 9$ (Observation 4.4): answering 0 becomes better.
9. Doing both only doubles it (Observation 5.3): they do not add.
10. Without a choice of points, one hand is missing (Observation 5.2): this divides practice.

Closing: Approximate $f(x) = 1/(1 + 25x^2)$ on equispaced points of $[-1, 1]$ and the polynomial gets worse as points are added while the spline on the very same points gets better——at $n = 29$, 1131.420474 against 0.001039, six

orders of magnitude out of one sample. Since f never exceeds 1, the polynomial is swinging to a thousand times the height of what it approximates. At $n = 5$ the two are only 1.38 apart—tried at few points, the difference is invisible. And there are two cures, either of which alone repairs it—move only the nodes to Chebyshev and the error is **0.00365254** (an improvement of 309762.9); move only the family to splines and it is **0.00103905** (an improvement of 1088894.0). The ratio between the two answers is 3.5152: equally good, not equal—two separate mechanisms landing in neighbouring decades. The cause of the collapse does not lie on the side of the function—the Lebesgue constant, computed from the node positions alone, is 1803783.3902 for equispaced against 3.0829 for Chebyshev at $n = 29$, and f does not enter that calculation at all. The equispaced polynomial passes an error of 1 at $n = 9$, beyond which answering 0 and doing nothing is the better policy. One thing separates them—whether the remedy changed where the samples are taken or which set of functions is fitted. If you choose your own sample points Chebyshev is available; if you do not, only the spline remains—against data already taken, one of the two hands is missing from the start. Under the criterion of Paper 300 all three rows point to distinct roots—one phrase names two things that do not meet. One last thing—taking more samples is not the cause of this. More samples are welcome. The cause is having demanded that one single polynomial pass through all of them. The data were not bad; the demand was too strong.

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