

The Wigner Angle Is an Area

— Velocity Space Is Hyperbolic: Where Poincaré's Geometry Works Inside Special Relativity — [Paper 33]

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— *From thinking to syncing.* —

ABSTRACT

To say that “in relativity velocities do not simply add” makes the matter sound smaller than it is. This paper shows in numbers that the reason they do not add is the curvature of velocity space. **NO NEW PHYSICAL LAW IS CLAIMED** — nothing here contradicts the Lorentz transformation. First, along one line the rapidities add exactly. Composing **0.5** with **0.5** gives **0.8**, **0.8** with **0.6** gives **0.9459459459459459**, and **0.99** with **0.99** gives **0.9999494975001263**: Einstein's rule and $\tanh(\theta_1 + \theta_2)$ agree to order 10^{-41} (computed here) (Section 2). Second, off that line a rotation is left over. Composing $\theta_1 = 1.0$ with $\theta_2 = 0.9$ at an angle α gives not a pure boost but a **boost times a rotation** (Section 3). Third, and this is the core. That rotation angle is the area of the rapidity triangle. Sweeping α from 15° to 170° , the Wigner angle and the angle defect of the hyperbolic triangle agree to order 10^{-40} (computed here). Gauss–Bonnet, having measured the ground, now works in velocity space (Section 4). Fourth, the separator is **non-collinearity**. At $\alpha = 0$ the triangle collapses to a segment and both area and rotation are exactly **0**; at $\alpha = 0.001^\circ$ the value is $5.69524130474 \times 10^{-6}$ and at 1° it is **0.005695078292** (Section 5). Fifth, the brake. “They add differently” is **not enough**. The composition is **neither commutative nor associative**: for $u = (0.8, 0)$ and $v = (0, 0.6)$, $u \oplus v = (0.8, 0.36)$ while $v \oplus u = (0.64, 0.6)$, the speeds agreeing and the directions not. Associativity fails by **0.0600488** (computed here) (Section 6). Sixth, the $1/2$ of the low-speed limit is not an adjustment but the leading term of an area (Section 7).

Scope of this paper: **NO NEW PHYSICAL LAW IS CLAIMED** — nothing here contradicts the Lorentz transformation. The hyperbolic character of velocity space, rapidity, the Thomas–Wigner rotation and gyrogroups are **all standard**. What is added is **an independent numerical verification that the Wigner angle equals the hyperbolic area**, naming the separator as **non-collinearity**, writing addresses for three “additions”, and noting that the same Gauss–Bonnet now works a second time inside the corpus.

1. Notation and position

Set $c = 1$ and define the rapidity of a velocity v as $\theta = \operatorname{artanh} v$. By velocity space is meant the unit hyperboloid $\{u : u \cdot u = 1, u^0 > 0\}$ with its induced metric, whose curvature is -1 (restoring c , $-1/c^2$).

Position: Paper 108 separated the three geometries of relativity by root, and Paper 91 treated intrinsic curvature. **This paper stands where the two meet: the set of velocities is itself a space with curvature. And the same Gauss–Bonnet used for the earth's surface (Paper 27) works here a second time.**

2. Along one line, the rapidities add exactly

Verification 2.1

v_1	v_2	Einstein sum	$\tanh(\theta_1 + \theta_2)$	difference
0.5	0.5	0.8	0.8	0.0
0.8	0.6	0.9459459459459459	0.9459459459459459	1.148×10^{-41}
0.99	0.99	0.9999494975001263	0.9999494975001263	0.0
0.1	0.2	0.2941176470588235	0.2941176470588235	5.74×10^{-42}

(QED)

Observation 2.2

Along one line rapidity is an additive coordinate and velocity is not. The rapidity is the hyperbolic distance from rest — lengths along a geodesic are being added, so of course they add.

3. Off that line, a rotation is left over

Verification 3.1: Compose $\theta_1 = 1.0$ along x with $\theta_2 = 0.9$ at an angle α .

α	composed rapidity	Wigner angle ω
0°	1.9	0.0
30°	1.849219714502761	0.1664176186094826
60°	1.694782265369292	0.3053032447581819
90°	1.431199273508851	0.3851025270775771
120°	1.053494636278679	0.3698915743144614
150°	0.5695508468232462	0.2335068067413657

(QED)

Observation 3.2

Two boosts do not compose to a boost. What remains is a rotation.

4. The core: that rotation angle is an area

Take three points on the unit hyperboloid: O (rest), P_1 (after the first boost) and P_2 (after both). The three sides have lengths $\theta_1, \theta_2, \theta_3$. On a surface of curvature -1 , Gauss–Bonnet gives **area** = $\pi -$ (**sum of the three angles**).

Verification 4.1

α	Wigner angle	hyperbolic area	difference
15°	0.0848772541736125232	0.0848772541736125232	2.612×10^{-40}
30°	0.166417618609482628	0.166417618609482628	1.894×10^{-40}
45°	0.241142741675334008	0.241142741675334008	2.009×10^{-40}
60°	0.30530324475818192	0.30530324475818192	6.888×10^{-41}
90°	0.38510252707757708	0.38510252707757708	5.166×10^{-41}
120°	0.369891574314461439	0.369891574314461439	2.87×10^{-41}
150°	0.233506806741365669	0.233506806741365669	9.184×10^{-41}
170°	0.0837525550248853798	0.0837525550248853798	1.521×10^{-40}

The worst difference across the table was 2.612×10^{-40} . (QED)

Observation 4.2

The leftover rotation is not an effect bolted on afterwards. It is precisely the angle defect of a triangle in a curved space.

Observation 4.3

The same Gauss–Bonnet is working a second time inside the corpus (the first was the earth's surface, Paper 27). The first time positive curvature gave an excess; here negative curvature gives a defect. The theorem is the same and only the sign differs.

Observation 4.4

So the Thomas–Wigner rotation is not a strange phenomenon peculiar to relativity. It is the holonomy of a closed circuit in a curved space — as a theorem it sits where Foucault's pendulum turning on a sphere sits.

5. The separator is non-collinearity

Verification 5.1

α	Wigner angle	area
0°	0.0	0.0
0.001°	$5.69524130474 \times 10^{-6}$	$5.69524130474 \times 10^{-6}$
0.1°	0.000569523967479	0.000569523967479
1°	0.005695078292	0.005695078292
10°	0.0567892433037	0.0567892433037

(QED)

Observation 5.2

At $\alpha = 0$ the triangle collapses to a segment. The area is 0, the rotation is 0, and the rapidities simply add. Move the angle and the curvature begins to bite — what must be moved is non-collinearity.

Observation 5.3

Section 2 could say “they add” not because velocities are special there, but because a one-dimensional submanifold has no area.

6. The brake: neither commutative nor associative

Verification 6.1: Take $u = (0.8, 0)$, $v = (0, 0.6)$ and $w = (0.4, 0.4)$.

$u \oplus v$	(0.8, 0.36)
$v \oplus u$	(0.64, 0.6)
$ u \oplus v $	0.8772684879784524
$ v \oplus u $	0.8772684879784524
$(u \oplus v) \oplus w$	(0.84891448825875, 0.45414266725742)
$u \oplus (v \oplus w)$	(0.87700534759358, 0.40106951871658)
failure of associativity	0.0600488

(QED)

Observation 6.2

The speeds agree and only the directions differ. The discrepancy is exactly the Wigner rotation.

Observation 6.3

To say “in relativity velocities add differently” makes the matter sound smaller than it is. Under this operation the set of velocities is not even a group. The accurate statement is that the failures appeared because something that is not a group was being written as one.

7. The 1/2 of the low-speed limit

Verification 7.1: At $\alpha = 90^\circ$ with both speeds equal to v ,

v	Wigner angle	$\frac{1}{2}v^2$	ratio
0.1	0.0050251467773656	0.005	1.00502935547
0.01	$5.0002500145843 \times 10^{-5}$	5.0×10^{-5}	1.00005000292
0.001	$5.0000025000015 \times 10^{-7}$	5.0×10^{-7}	1.0000005
0.0001	$5.000000025 \times 10^{-9}$	5.0×10^{-9}	1.000000005

(QED)

Observation 7.2

The ratio converges to 1. The $1/2$ carried by the leading term is the same $1/2$ that appears in the spin–orbit coupling. It is not a fudge factor but the leading term of an area — for the same reason that the area of a triangle begins as half the product of two sides.

8. The census

Observation 8.1

“addition”	space	curvature	the operation
Galilean velocity	Euclidean	0	an abelian group
rapidity	geodesic length in hyperbolic space	−1	additive only along a line
relativistic velocity	points of that same space	−1	neither commutative nor associative

Observation 8.2

All three have been called adding velocities. The separator is the curvature of the space the set lives on.

9. Conclusion

1. Along one line the rapidities add exactly (Ver. 2.1): differences of order 10^{-41} .
2. Off that line a rotation is left over (Ver. 3.1).
3. That rotation angle is the area of the triangle (Ver. 4.1): worst difference 2.612×10^{-40} .
4. The separator is non-collinearity (Ver. 5.1): exactly 0 at $\alpha = 0$.
5. The composition is neither commutative nor associative (Ver. 6.1): a failure of 0.0600488.
6. The $1/2$ of the low-speed limit is the leading term of an area (Ver. 7.1).

Closing: Velocities do not add plainly, and not because the rule is odd. It is because the place where velocities live is curved. And in a curved place, going once around a triangle turns you — on the ground and in velocity space alike, one theorem is saying one thing.

References

1. Poincaré, H. (1906). *Sur la dynamique de l'électron*. Rendiconti del Circolo Matematico di Palermo, 21.
2. Varičák, V. (1910). *Anwendung der Lobatschewskischen Geometrie in der Relativtheorie*. Physikalische Zeitschrift, 11.
3. Thomas, L. H. (1926). *The motion of the spinning electron*. Nature, 117.
4. Wigner, E. P. (1939). *On unitary representations of the inhomogeneous Lorentz group*. Annals of Mathematics, 40(1).
5. Aravind, P. K. (1997). *The Wigner angle as an anholonomy in rapidity space*. American Journal of Physics, 65(7).
6. Ungar, A. A. (1988). *Thomas rotation and the parametrization of the Lorentz transformation group*. Foundations of Physics Letters, 1(1).
7. Yamagishi, Y. (2026). *Why Does a “Maximum Speed” Exist?* [Paper 83]. Zenodo. 10.5281/zenodo.21741190
8. Yamagishi, Y. (2026). *Curvature Is Intrinsic* [Paper 91]. Zenodo. 10.5281/zenodo.21758406
9. Yamagishi, Y. (2026). *The Geometry Where the Invariant Lives* [Paper 108]. Zenodo. 10.5281/zenodo.21837313
10. Yamagishi, Y. (2026). *How Far Does Light Determine Geometry?* [Paper 111]. Zenodo. 10.5281/zenodo.21961685
11. Yamagishi, Y. (2026). *What Differs Between the Twins Is Not a Lagging Clock but the Length of a Worldline* [Paper 123]. Zenodo. 10.5281/zenodo.22055231

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