

Area, a Continuous Quantity, Comes Out Exactly from Counting Two Kinds of Points

— Interior Points Count **1**, Boundary Points **$1/2$** , and Then **-1** — That **-1** Is Not the Shape but One Full Turn
— In Three Dimensions the Volume Grows Tenfold While the Point Count Stays at **4** — [Paper 331]

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September 2026

— *From thinking to syncing.* —

Abstract

The area of a polygon whose vertices sit on lattice points comes out exactly from counting points. This paper asks why a count can produce a continuous quantity—the answer is that the three terms on the right count three different things. No new mathematical theorem and no new law is claimed. First, for a concave hexagon $A = 12.0$, $B = 16$, $I = 5$ — and $5 + 8 - 1 = 12$ (Section 2). Second, all five polygons agreed. Including one of area 10 with no interior points and one of area 17.5 with five boundary points (Section 2). Third, this is the core of the paper. The -1 is not the shape; it is one full turn (Section 3). Fourth, move a vertex off the lattice and it breaks. Against a true area of 0.25 the formula returns 0.0 (Section 4). Fifth, no such formula exists in three dimensions. The volume runs from 0.1667 to 1.6667 while the lattice points stay at 4 (Section 4). Sixth, the separator is whether the boundary is one-dimensional or two (Section 5).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed——Pick’s theorem of 1899, Reeve’s tetrahedron, the shoelace formula, and the sum of interior angles are all standard. We do not build a theory of lattices——only the integer lattice $\mathbb{Z}^2$ is used; general lattices are not entered. We do not treat Ehrhart polynomials——the general count of lattice points in a k -fold dilate is not entered. It is mentioned and no more. We prove no theorem——what is done here is to show, by actually summing angles, what each of the three terms on the right counts. No proof of Pick’s theorem itself is given. We do not treat regions with holes——only simply connected simple polygons are looked at. We do not treat number theory——where lattice-point counting joins arithmetic is not entered. Relation to earlier papers: Paper 149 showed that what fixes the denominator is not the determinant but the largest invariant factor——there too lattice points are counted, to confirm the volume of a fundamental domain. But 149 measures an index of the lattice itself, while this paper measures the area of a figure placed on it. Paper 316 showed the same integer returning from three different counts——here one continuous quantity is returned by two kinds of count. The pair runs the other way. Paper 205 showed the shortest enclosure is not always a disc——it too asks after the relation of area to boundary. Paper 300 showed whether two things share a root is decidable——by that test the two-dimensional and three-dimensional cases here have distinct roots. Paper 182 showed “the curse of dimensionality” is not one curse——what is lost here as the dimension rises is the existence of the formula itself. What is added is checking five polygons, two of them concave, against exhaustive recounts, deriving the -1 numerically from the sum of interior angles, setting side by side a case where the hypothesis is broken and a case where the formula ceases to exist, and putting the separator on the dimension of the boundary.

1. Notation and the Question

Setting: On the integer lattice $\mathbb{Z}^2$ place a simple polygon all of whose vertices are lattice points. Three quantities are counted.

Symbol	What is counted
A	the area (a continuous quantity)
B	the number of lattice points on the boundary
I	the number of lattice points in the interior

Observation 1.1

A is continuous, while B and I are integers. Producing a real number from integers usually takes a limit or an integral——yet here it comes out exactly from counting finitely many things.

Observation 1.2

The way of measuring is fixed first. A from the shoelace formula, B as the sum of $\gcd(|\Delta x|, |\Delta y|)$ over the edges, I by exhaustive search—three separate methods, then compared.

Question: Why can a count produce a continuous quantity, and how far does it reach?

2. Five Polygons, Not One Miss

Calculation 2.1: For the concave hexagon $(0, 0), (4, 0), (4, 2), (2, 2), (2, 4), (0, 4)$ (computed in this paper).

Quantity	Value
Area A by the shoelace formula	12.0000
Boundary lattice points B (sum of $\gcd$)	16
Interior lattice points I (exhaustive)	5
$I + B/2 - 1$	$5 + 8 - 1 = 12.0000$

Observation 2.2

A check—the boundary points were also counted exhaustively, giving 16, in agreement with the sum of $\gcd$. Two independent counts return the same integer before the table is read.

Calculation 2.3: One example proves nothing. Four more of different shape (computed in this paper).

Polygon	A	B	I	$I + B/2 - 1$
Right triangle $(0, 0), (5, 0), (0, 3)$	7.5	9	4	7.5
Oblique triangle $(0, 0), (3, 1), (1, 3)$	4.0	4	3	4.0
Convex quadrilateral $(0, 0), (6, 2), (5, 5), (1, 4)$	17.5	5	16	17.5
U-shape (concave octagon)	10.0	22	0	10.0

Observation 2.4

The last two rows are the two extremes. The quadrilateral has only 5 boundary points yet an area of 17.5; the U-shape has no interior points at all yet an area of 10——neither count alone gives the area. Both kinds are needed.

Observation 2.5

The oblique triangle has not one edge parallel to an axis. It still comes out at 4.0——orientation does not enter the formula.

Observation 2.6

Concavity is no obstacle. Neither the hexagon nor the U-shape is convex——convexity is not among the hypotheses.

3. The Core: The Three Terms Count Three Different Things

Setting: Give each lattice point a weight: the fraction of a full turn the polygon occupies around that point. An interior point occupies the whole turn and weighs 1; a point on an edge but not a vertex occupies half and weighs 1/2; a vertex weighs its interior angle θ_i over 2π .

Calculation 3.1: Sum the interior angles at the vertices by actually adding the angles (computed in this paper).

Polygon	Vertices v	Sum of interior angles	$(v - 2)\pi$
Concave hexagon	6	4.000000π	4π
Right triangle	3	1.000000π	1π
Oblique triangle	3	1.000000π	1π
Convex quadrilateral	4	2.000000π	2π
U-shape (concave)	8	6.000000π	6π

Observation 3.2

It is $(v - 2)\pi$ even when concave. All five agreed, two of them concave—the sum of interior angles is fixed by the number of vertices and not by the shape.

Observation 3.3

The vertices therefore contribute $(v - 2)\pi/(2\pi) = v/2 - 1$, and the weights sum to

$$I + \frac{B - v}{2} + \left(\frac{v}{2} - 1\right) = I + \frac{B}{2} - 1.$$

Observation 3.4

This is the core of the paper. The three terms are not one thing counted three times— I counts points occupying a whole turn, $B/2$ points occupying half a turn, and -1 is one full turn. Only the last is not a count of points.

Observation 3.5

What the -1 really is has nothing to do with the shape. It does not change with v , with concavity, or with oblique edges—it comes from nothing but going once around and returning to the same heading.

Observation 3.6

The act of subtracting an overlap is what sits here. A boundary point belongs to the inside and the outside alike and so is counted half; a vertex is counted by its angle—the right-hand side is assembled by taking back what was counted twice.

Observation 3.7

Where this stands in the corpus.The phrase “Euler characteristic” occurs in 28 papers of this corpus; “inclusion-exclusion” occurs in none—the roof has been handled again and again and the foundation has never once been named. This paper lays that foundation down in the concrete form of area.

4. How Far It Reaches — Two Ways of Breaking

Setting: Drop the hypotheses one at a time and see what happens.

Calculation 4.1: Move a vertex off the lattice. Take the triangle $(0, 0), (1, 0), (0.5, 0.5)$ (computed in this paper).

Quantity	Value
True area by the shoelace formula	0.2500
Boundary lattice points B	2
Interior lattice points I	0
$I + B/2 - 1$	0.0000

Observation 4.2

0.25 becomes 0.It does not break by drifting a little—it returns zero area for a figure that has area. That the vertices be lattice points is not decoration.

Calculation 4.3: Raise the dimension by one. Take the tetrahedron $(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, k)$ (computed in this paper).

k	Volume = $k/6$	Total lattice points (boundary and interior)
1	0.1667	4
2	0.3333	4
3	0.5000	4
5	0.8333	4
10	1.6667	4

Observation 4.4

Every vertex is a lattice point; the hypothesis holds. And still the volume grows tenfold while the count does not move off 4—a function whose only input is the number of lattice points cannot tell these two tetrahedra apart.

Observation 4.5

This is not “the three-dimensional formula has not been found yet”. It would have to return different outputs for the same input—no such function can exist. That Pick’s theorem has no three-dimensional version is not a matter of how one looks.

Observation 4.6

The two failures are of different kinds. Moving a vertex off the lattice breaks a hypothesis; failing in three dimensions happens with every hypothesis intact—the first is a misuse, the second a limit of the theorem.

5. The Separator: Is the Boundary One-Dimensional or Two?

Observation 5.1

The separator—whether the points on the boundary can be written as finitely many “halves”.

	Dimension of boundary	Weight of a boundary point	Formula
Two dimensions	1 (a line)	exactly $1/2$	exists
Three dimensions	2 (a surface)	not constant	does not exist

Observation 5.2

In two dimensions every lattice point on an edge carries the same $1/2$. An edge is a straight line, so a point on it necessarily occupies half a turn—that constancy is why a finite count suffices.

Observation 5.3

In three dimensions a point on a face also carries $1/2$. And that is still not enough—Reeve’s tetrahedron has no point on a face and none inside. It has only its four vertices, and its volume can be made as large as one likes.

Observation 5.4

To be stated honestly—three dimensions do have a counting theory. The Ehrhart polynomial writes the number of lattice points in a k -fold dilate as a polynomial in k —but that gives one polynomial per figure; it is not a single formula taking a point count to a volume. It is the latter that this paper says does not exist.

Observation 5.5

Where this stands relative to Paper 182. Paper 182 wrote that “the curse of dimensionality” is not one curse—what is lost here as the dimension rises is neither cost nor accuracy but the existence of the formula. The curse has kinds still without names.

Observation 5.6

One last thing—this theorem is hardly ever used as a way of measuring area. The shoelace formula is faster and demands no hypotheses—what keeps this one alive is not that the answers agree, but that it shows two worlds pointing at the same number. The value is not what is worth having; the equals sign is.

6. The Census and the Close

1. $A = 12.0$, $B = 16$, $I = 5$ (Calculation 2.1): $5 + 8 - 1 = 12$.
2. $B = 16$ by two separate counts (Observation 2.2): sum of gcd and exhaustive.
3. All five polygons agree (Calculation 2.3): two of them concave.
4. Area 17.5 from 5 boundary points; area 10 from no interior points (Observation 2.4): neither alone.
5. Interior angles sum to $(v - 2)\pi$ (Calculation 3.1): concavity included.
6. The -1 is one full turn (Observation 3.5): unrelated to shape.
7. “Inclusion-exclusion” occurs in no paper of this corpus (Observation 3.7): “Euler characteristic” in 28.
8. Off the lattice, 0.25 becomes 0 (Calculation 4.1): not a small drift.
9. Tenfold volume, still 4 points (Calculation 4.3): Reeve’s tetrahedron.
10. The 3D version is not undiscovered but impossible (Observation 4.5): same input, different outputs.

Closing: For a polygon whose vertices are lattice points, area, a continuous quantity, comes out exactly from counting two kinds of points—for a concave hexagon $A = 12.0000$, $B = 16$, $I = 5$, and $5 + 8 - 1$ is exactly 12. Not one of five polygons of differing shape, two of them concave, missed. And both kinds of count are needed—the convex quadrilateral has only 5 boundary points yet an area of 17.5, and the U-shape has no interior points at all yet an area of 10. Neither count alone gives the area. The three terms on the right are not one thing counted three times— I counts points occupying a whole turn, $B/2$ points occupying half a turn, and -1 is not a count of points at all but one full turn. Summing the interior angles gives $(v - 2)\pi$ even for concave polygons, so the vertices contribute $v/2 - 1$. The -1 has nothing to do with the shape; it comes from nothing but going once around and returning to the same heading. There are two ways of breaking—move a vertex off the lattice and against a true area of 0.25 the formula returns 0.0; that breaks a hypothesis and is a misuse. But in three dimensions it fails with every hypothesis intact—the tetrahedron $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, k)$ keeps exactly 4 lattice points while its volume runs from 0.1667 to 1.6667. A function whose only input is a point count cannot tell the two apart. The three-dimensional version is not undiscovered; it cannot exist. One thing separates them—whether the points on the boundary can be written as finitely many halves. In two dimensions an edge is a straight line, so every lattice point on it occupies half a turn, always. In three dimensions that constancy is not enough, and volume grows with no point appearing on a

face or inside. Paper 182 wrote that “the curse of dimensionality” is not one curse——what is lost here as the dimension rises is neither cost nor accuracy but the existence of the formula itself. One last thing——this theorem is hardly ever used as a way of measuring area. The shoelace formula is faster and demands no hypotheses. What keeps this one alive is not that the answers agree, but that it shows two worlds pointing at the same number. The value is not what is worth having; the equals sign is.

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