

“Cannot Be Constructed” Is a Property of the Tools, Not of the Figure

— An Angle Resists Trisection Because $\cos 20^\circ = \mathbf{0.9396926208}$ Is a Root of $8c^3 - 6c - 1$, and Straightedge and Compass Do Not Reach Degree **3** — Fold Paper and They Do — [Paper 347]

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— *From thinking to syncing.* —

Abstract

An angle, it is said, cannot be trisected by construction. This paper asks what it is that makes it impossible—the answer is not the figure but the tools allowed. No new mathematical theorem and no new law is claimed. First, $\cos 20^\circ = \mathbf{0.9396926208}$ is a root of $8c^3 - 6c - 1 = 0$ (Section 2). Second, substituting leaves a residual of $\mathbf{8.882 \times 10^{-16}}$, within double-precision rounding (Section 2). Third, this is the core of the paper. Straightedge and compass build only extensions of degree a power of two, and **3** is not one (Section 3). Fourth, origami solves cubics, so the same angle can be trisected (Section 4). Fifth, doubling the cube, $2^{1/3} = \mathbf{1.2599210499}$, folds for the same reason (Section 4). Sixth, the separator is whether the permitted moves stop at degree **2** or reach **3** (Section 5).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed——Wantzel’s theorem of 1837, the triple-angle formula, Huzita’s origami axioms, and the degree of a field extension are all standard. We do not develop Galois theory——solvability criteria and group-theoretic treatment are not entered. Only one quantity, the degree of an extension, is used. We prove no impossibility——Wantzel’s theorem is quoted and not proved. It is mentioned and no more. We do not exhaust the origami axioms——the independence and completeness of Huzita’s seven are not entered. Only the fact that they reach degree 3 is used. We do not treat squaring the circle——the transcendence of π is another matter, and origami cannot do it either. We show no actual folds——diagrams of procedure are not the subject. Relation to earlier papers: Paper 179 showed “cannot be constructed” to have distinct roots——there, on the set-theoretic side, the presence or absence of the axiom of choice divided constructibility. Here, on the geometric side, the degree of the tools divides it. Under one phrase, at least two distinct roots. Paper 331 showed area coming out of counting two kinds of points, and no such formula existing in three dimensions——there dimension set the limit; here the tools do. Both name the reason for a “there is none”. Paper 324 showed that no search is universal because of how the problems are counted——there too impossibility came from the frame and not from the object. Paper 300 showed whether two things share a root is decidable——trisection and duplication share a root by that test: both meet the wall at degree 3. What is added is showing $\cos 20^\circ$ to be a root with a residual of 8.882×10^{-16} and confirming it independently by Newton’s method, placing $\cos 30^\circ$ as a counter-case that is not a root, confirming numerically that the polynomial follows from the triple-angle formula, and putting the separator on the degree of the permitted moves.

1. Notation and the Question

Setting: Trisect a given angle with straightedge and compass——a problem from the Greeks, shown impossible by Wantzel in 1837.

Observation 1.1

The word “impossible” can be read two ways. (a) The figure has no trisection point. (b) These tools cannot reach that point——the true reading is (b).

Observation 1.2

The trisection point of course exists. A third of 60° is 20° , and the point is there——a protractor will draw it. The question is whether straightedge and compass alone can reach it.

Observation 1.3

The way of measuring is fixed first. Take the trisection of 60° as the case, and ask whether $\cos 20^\circ$ is constructible—it is not that no angle can be trisected; 90° can.

Question: Is the impossibility in the figure or in the tools?

2. $\cos 20^\circ$ Is a Root of $8c^3 - 6c - 1$

Setting: The triple-angle formula is $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$. With $\theta = 20^\circ$ and $\cos 60^\circ = 1/2$,

$$4c^3 - 3c = \frac{1}{2}, \quad \text{that is} \quad 8c^3 - 6c - 1 = 0.$$

Calculation 2.1: Substitute $\cos 20^\circ$ (computed in this paper).

Quantity	Value
$\cos 20^\circ$	0.9396926208
Residual in $8c^3 - 6c - 1$	8.882×10^{-16}
Real root by Newton's method	0.9396926208
Difference between the two	0.000×10^0

Observation 2.2

Three checks. (a) The residual 8.882×10^{-16} lies within double-precision rounding. (b) Solving the polynomial independently by Newton's method returns a root agreeing with $\cos 20^\circ$ to the last digit. (c) Substituting into the triple-angle formula $4c^3 - 3c$ gives 0.500000000000, agreeing with $\cos 60^\circ = 1/2$ —three routes to the same number before going on.

Calculation 2.3: A counter-case. Is $\cos 30^\circ$ a root of the same polynomial (computed in this paper)?

Quantity	Value
$\cos 30^\circ$	0.8660254038
Residual in $8c^3 - 6c - 1$	1.000000

Observation 2.4

A residual of 1.000000; not a root. Fifteen orders of magnitude between 8.882×10^{-16} and 1.000000—the polynomial is specific to 20° and does not accept any angle. Confirmation that the check is not one that everything passes.

Observation 2.5

And the polynomial is irreducible over the rationals. The rational root theorem leaves the candidates $\pm 1, \pm 1/2, \pm 1/4, \pm 1/8$, and all eight give a non-zero value—so the minimal polynomial of $\cos 20^\circ$ has degree 3.

3. The Core: Straightedge and Compass Build Only Powers of Two

Observation 3.1

The moves available are the intersection of two lines, of a line and a circle, and of two circles. Each, solved simultaneously, gives an equation of degree 1 or 2—so each move multiplies the degree of the field extension by 1 or by 2.

Observation 3.2

This is the core of the paper. However many moves are stacked, the reachable degree is a power of two—2, 4, 8, 16, and never 3.

Observation 3.3

The minimal polynomial of $\cos 20^\circ$ has degree 3. And 3 is not a power of two——so it cannot be reached. That is the skeleton of Wantzel’s argument.

Observation 3.4

No property of the figure enters this. The angle is not peculiarly difficult and 20° is not pathological——it is settled by one arithmetic fact, that 3 is not a power of two.

Observation 3.5

So some angles can be trisected with the same tools. A third of 90° is 30° , and $\cos 30^\circ = \sqrt{3}/2$ has degree 2——“the trisection of an angle is impossible” means, exactly, “impossible for a general angle”, and particular angles yield.

Observation 3.6

It pairs with Paper 179. Paper 179 wrote that “cannot be constructed” has distinct roots, dividing on the axiom of choice in set theory——here it divides on the degree of the tools in geometry. Under one phrase, at least two distinct roots.

4. Origami Reaches Degree 3

Setting: Folding paper has axioms too (Huzita’s seven). The sixth permits a fold placing two points simultaneously onto two lines——and that move solves a cubic.

Observation 4.1

So origami reaches $\cos 20^\circ$. The same $8c^3 - 6c - 1 = 0$, solved by one crease——what was impossible with straightedge and compass becomes possible with paper.

Calculation 4.2: The same happens for doubling the cube (computed in this paper).

Problem	Number to reach	Degree of the minimal polynomial
Trisecting 60°	$\cos 20^\circ = 0.9396926208$	3
Doubling the cube	$2^{1/3} = 1.2599210499$	3

Observation 4.3

A check—the cube of $2^{1/3}$ is 2.000000000000. An independently verifiable relation confirms the value before it is used.

Observation 4.4

Two classical problems were impossible for one and the same reason. Both have minimal polynomials of degree 3—by the criterion of Paper 300, trisection and duplication share a root. Two problems of different appearance were meeting one wall.

Observation 4.5

And for the same reason both yield to origami. If the wall is one, the tool that clears it is one—sharing a root shows itself as sharing a solution.

Observation 4.6

To be stated honestly—origami cannot square the circle either. π is transcendental and lies in no finite algebraic extension—reaching degree 3 does not reach a transcendental. There is another wall that strengthening the tools does not clear.

5. The Separator: Do the Permitted Moves Stop at Degree 2 or Reach 3?

Observation 5.1

The separator—the degree of the equation one move can solve: up to 2, or reaching 3.

Tools	Degree per move	Reachable extension degrees	Trisection, duplication
Straightedge and compass	up to 2	powers of two only	impossible
Origami (axiom 6)	reaches 3	includes 3	possible

Observation 5.2

The separator is not on the side of the problem. The same angle, the same paper, the same number——what makes the difference is only what counts as one move.

Observation 5.3

So “impossible” must always be said with its tools. “The trisection of an angle is impossible” is a statement about straightedge and compass, not about angles——mistake the subject and one believes in an impossibility that does not exist.

Observation 5.4

The separator can be moved. Two marks on the straightedge (a neusis construction) already reach degree 3——Archimedes trisected with exactly that. Loosen the tools by one notch and the wall is gone.

Observation 5.5

Where this stands relative to Paper 324. Paper 324 wrote that no search is universal because of how the problems are counted——there too the impossibility came from the frame and not from the object. Here the frame is the toolkit.

Observation 5.6

One last thing——that this problem stood for two thousand years is not because the question was hard. That the answer is “it cannot be done” and that this “cannot” is a property of the tools are two different understandings——what Wantzel did was not to attempt the trisection but to measure the tools.

6. The Census and the Close

1. $\cos 20^\circ = 0.9396926208$ (Calculation 2.1): **a root of $8c^3 - 6c - 1$.**
2. Residual 8.882×10^{-16} (Calculation 2.1): **within double-precision rounding.**
3. Newton's root differs by 0 (Calculation 2.1): **agreement by an independent method.**
4. Triple angle gives 0.500000000000 (Observation 2.2): **agreeing with $\cos 60^\circ$.**
5. $\cos 30^\circ$ leaves a residual of 1.000000 (Calculation 2.3): **not a check everything passes.**
6. Straightedge and compass give powers of two (Observation 3.2): **never 3.**
7. No property of the figure enters (Observation 3.4): **only that 3 is not a power of two.**
8. 90° can be trisected (Observation 3.5): **"for a general angle" is the exact claim.**
9. $2^{1/3} = 1.2599210499$ is also degree 3 (Calculation 4.2): **two problems, one root.**
10. Origami cannot square the circle (Observation 4.6): **another wall remains.**

Closing: An angle, it is said, cannot be trisected by construction, and the word "impossible" can be read two ways —(a) the figure has no trisection point, (b) these tools cannot reach it. The true reading is (b). A third of 60° is 20° , and the point is there. From the triple-angle formula, $\cos 20^\circ$ is a root of $8c^3 - 6c - 1 = 0$ —substituting **0.9396926208** leaves **8.882×10^{-16}** , within double-precision rounding. Solving the polynomial independently by Newton's method returns a root agreeing to the last digit, and substituting into $4c^3 - 3c$ gives 0.500000000000, agreeing with $\cos 60^\circ = 1/2$. As a counter-case, $\cos 30^\circ$ leaves a residual of 1.000000: the polynomial is specific to 20° . And that polynomial is irreducible over the rationals, so the minimal polynomial has degree 3 —whereas straightedge and compass offer only the intersections of two lines, a line and a circle, and two circles, each of degree 1 or 2. However many moves are stacked, the reachable degree is a power of two, and 3 never appears. That is the skeleton of Wantzel's argument, and no property of the figure enters it — it is settled by the arithmetic fact that 3 is not a power of two. So some angles do yield to the same tools — a third of 90° is 30° , and $\cos 30^\circ = \sqrt{3}/2$ has degree 2. "The trisection of an angle is impossible" means exactly "impossible for a general angle". Origami reaches degree 3 — Huzita's sixth axiom permits a fold placing two points onto two lines, which solves a cubic. The same $8c^3 - 6c - 1 = 0$ falls to one crease. Doubling the cube, $2^{1/3} = 1.2599210499$, also has a minimal polynomial of degree 3, so two classical problems were meeting one wall — by the criterion of Paper 300 they share a root, and where the wall is one the tool that clears it is one. One thing separates them — the degree of the equation one move can solve: up to 2, or reaching 3. The separator is not on the side of the problem — the same angle, paper and number, and what differs is only what counts as one move. So "impossible" must always be said with its tools: the claim is about straightedge and compass, not about angles. And it can be moved — two marks on the straightedge already reach degree 3, and Archimedes trisected with exactly that. To be said honestly — origami cannot square the circle either. π is transcendental and lies in no finite algebraic extension. Reaching degree 3 does not reach a transcendental, and there is another wall that strengthening the tools does not clear. One last thing — that this

problem stood for two thousand years is not because the question was hard. That the answer is “it cannot be done” and that this “cannot” is a property of the tools are two different understandings. What Wantzel did was not to attempt the trisection but to measure the tools.

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