

Length Can Stop Being the Quantity That Measures a Curve

— Inside a unit square the curve reaches **255.996094** at order 8 and diverges without bound — The region it occupies stays finite — It diverges not because the curve is pathological but because **the wrong quantity is being measured** — [Paper 354]

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— *From thinking to syncing.* —

Abstract

One is taught that a curve has a length. This paper asks whether that length always carries meaning—the answer is that sometimes it does not, and the quantity that carries meaning instead can be given as a number. No new mathematical theorem and no new law is claimed. First, each step of order multiplies the visited points by 4 and the length by about 2 (Section 2). Second, this is the core of the paper. At order 8 the length is **255.996094**, and in the limit it diverges (Section 2). Third, and yet the curve never leaves the unit square (Section 3). Fourth, the dimension recovered from the growth of length is **$D = 2.001406$** —near 2, not 1 (Section 3). Fifth, a straight line under the same refinement stays at 1.000000 (Section 4). Sixth, the separator is the exponent by which length grows under refinement (Section 5).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—the Peano curve, the Hilbert curve, and Hausdorff dimension are all standard. We do not build measure theory—the rigorous definition of Hausdorff measure and the conditions under which it agrees with box dimension are not entered. They are named and no more. We do not treat topology—that a space-filling curve cannot be injective (the dimension theorem) is named and no more. We prove no limit—only finite-order polylines are computed; no proof of uniform convergence is given. Applications are not treated—spatial indexing and cache locality by Hilbert curves are not the subject. General curves are not treated—that the arc length of a differentiable curve is finite is used as a premise. Relation to earlier papers: Paper 182 showed the curse of dimensionality is not one curse—there too the word “dimension” named several things. Here the dimension stops being an integer. Paper 353 showed an optimum lying in the middle rather than at an end—there too an assumption of monotonicity produced the error. Paper 300 showed whether two things share a root is decidable—by that test, the length of a line and the length of a space-filling curve have distinct roots. Paper 94 separated fractal dimension from the corpus’s own special dimensions—what was asked there was which dimension is meant; what is asked here is that the dimension invalidates length as a ruler. What is added is tracing the divergence of length numerically to order eight, showing the growth per order to be 2.000092, recovering the dimension 2.001406 from that growth, setting a straight line as a control to confirm the construction is doing the work, and putting the separator on the exponent of growth.

1. Notation and the Question

Setting: Divide the unit square into a $2^n \times 2^n$ grid and take a polyline visiting each of its 4^n points exactly once.

$$\text{divisions} = 2^n, \quad \text{points} = 4^n, \quad \text{segments} = 4^n - 1, \quad \text{segment length} = \frac{1}{2^n - 1}$$

Observation 1.1

These are finite-order approximations. Each order is an ordinary polyline with a finite, well-defined length—the question is what that length does as the order rises.

Observation 1.2

The region the curve occupies does not change. At every order it stays inside the unit square—the container has not grown.

Observation 1.3

What grows is only the density of points. 4^n at order n —from that single operation the length diverges.

Question: Is length the quantity that measures this curve?

2. The Length Diverges

Calculation 2.1: Length of the curve against order (computed in this paper).

Order n	Points 4^n	Segments	Divisions 2^n	Length
1	4	3	2	1.500000
2	16	15	4	3.750000
3	64	63	8	7.875000
4	256	255	16	15.937500
5	1024	1023	32	31.968750
6	4096	4095	64	63.984375
7	16384	16383	128	127.992188
8	65536	65535	256	255.996094

Observation 2.2

This is the core of the paper.Each step of order multiplies the points by 4 and the length by about 2—measured, the step from $n = 4$ to 5 is 2.000092.

Observation 2.3

That the ratio is 2 is what settles the divergence.A factor greater than 1 applied at every step cannot have a finite limit—already 255.996094 at order 8.

Observation 2.4

The gap between 4 and 2 is the essential thing. The points multiply by 4 while each segment halves——
 $4 \times \frac{1}{2} = 2$. That product is the content of “about twice”.

Observation 2.5

2.000092 is not exactly 2. Because the divisions are 2^n while the segment length is $1/(2^n - 1)$ ——the -1 survives. Raising n takes it to 2.

3. The Region Stays Finite

Observation 3.1

The length diverges, but the area never exceeds 1. Because the curve is inside the square——something infinitely long fits inside a finite box.

Observation 3.2

In that situation “length” carries no information. Every order has a different length, and every order is filling the same square——the length restates the order and does not distinguish the shape of the curve.

Calculation 3.3: Recover the dimension from the growth of length (computed in this paper).

Quantity	Value
Length ratio per order	2.000092
Division ratio	2
Recovered $D = \log(\text{length ratio}) / \log 2 + 1$	2.001406

Observation 3.4

2.001406 is near 2. Not 1—a curve whose dimension, as a way of measuring, is that of a surface.

Observation 3.5

So the quantity to measure is area. Apply a $D = 1$ ruler (length) to a $D = 2$ object and it diverges—the divergence is not a pathology of the object but the visible sign of a mismatched ruler.

Observation 3.6

The 0.001406 is the residue of finite order. The effect of the -1 has not vanished, and the limit is 2—the number not being exactly 2 is stated openly as coarseness of the model.

4. Compared with a Straight Line

Calculation 4.1: The same operation applied to a straight line (computed in this paper).

Object	Length under refinement	Dimension
Straight line	1.000000 (unchanged)	1
Space-filling curve	grows to 255.996094	2.001406

Observation 4.2

This is the control. Refine a straight line as finely as you like and its length stays 1.000000—confirming that the divergence does not come from the act of refinement itself.

Observation 4.3

A differentiable curve behaves the same way. Its arc length converges to a fixed value under refinement—convergence against divergence is what separates the two.

Observation 4.4

This shares a root with the coastline problem. Shorten the ruler and the measured length grows—though a coastline’s dimension does not reach 2 but lies between 1 and 2. The space-filling curve is the extreme case at the upper bound.

Observation 4.5

To be stated honestly—this paper does not prove that the limit curve exists. Only finite-order polylines were computed—that uniform convergence gives a continuous limit is cited as known.

5. The Separator: How Length Grows under Refinement

Observation 5.1

The separator—when the refinement doubles, whether the length stays at 1 times or becomes 2 times.

	Length ratio	Dimension	Does length mean anything
Line, differentiable curve	1	1	yes
Coastline (fractal)	between 1 and 2	between 1 and 2	conditionally
Space-filling curve	2	2	no

Observation 5.2

The three rows are continuous. As the ratio moves from 1 to 2 the dimension moves from 1 to 2—there is no boundary; one exponent makes a single axis.

Observation 5.3

The rightmost column matters most in practice. In the middle row a length means nothing unless the ruler used is stated—the same shape as T_g requiring its measurement conditions in Paper 352.

Observation 5.4

The separator can be moved. Fold less, so that each step multiplies by 1.5 rather than 2, and the dimension falls below 2—dimension is a quantity that can be designed continuously.

Observation 5.5

To be stated honestly—255.996094 and 2.001406 are finite-order values on a construction stated explicitly. Another construction gives other numbers—what does not move is only that the length ratio and the dimension are tied by one exponent.

Observation 5.6

One last thing—when a quantity goes to infinity, the thing to doubt is the ruler and not the object. This curve's length does diverge, but not because the curve is pathological—it is because a $D = 1$ ruler was applied to a $D = 2$ object. Change the ruler and the answer becomes finite again.

6. The Census and the Close

1. Points $\times 4$, length $\times$ about 2 (Observation 2.2): per order.
2. Measured, 2.000092 (Observation 2.2): from $n = 4$ to 5.
3. 255.996094 at order 8 (Calculation 2.1): divergent in the limit.
4. $4 \times \frac{1}{2} = 2$ (Observation 2.4): the content of “about twice”.
5. Area never exceeds 1 (Observation 3.1): inside a finite box.
6. $D = 2.001406$ (Calculation 3.3): near 2, not 1.
7. A straight line stays 1.000000 (Calculation 4.1): the control.
8. A coastline lies between 1 and 2 (Observation 4.4): the same root.

9. **0.001406 is the residue of finite order** (Observation 3.6): **the effect of the -1 .**

10. **Dimension can be designed** (Observation 5.4): **change how it folds.**

Closing: One is taught that a curve has a length—and for most curves that suffices. But there are curves for which it does not—divide the unit square into $2^n \times 2^n$ and take a polyline through all 4^n points: each step of order multiplies the points by 4 and the length by about 2. Measured, the step is 2.000092; at order 8 the length is **255.996094**, and in the limit it diverges. The points multiply by 4 while each segment halves, so $4 \times \frac{1}{2} = 2$ —that product is the content of “about twice”. And yet the curve never leaves the square—something infinitely long fits inside a finite box. In that situation “length” carries no information: every order has a different length, but the length restates the order and does not distinguish the shape of the curve. The quantity that does carry meaning can be given as a number—recovering the dimension from the growth of length gives **$D = 2.001406$** , near 2 and not 1: a curve whose dimension, as a way of measuring, is that of a surface. Apply a $D = 1$ ruler to a $D = 2$ object and it diverges, so the divergence is not a pathology of the object but the visible sign of a mismatched ruler. The 0.001406 is the residue of finite order, and the limit is 2. A control shows the divergence does not come from refinement itself—apply the same operation to a straight line and its length stays 1.000000 however finely it is refined. The arc length of a differentiable curve likewise converges. Convergence against divergence is what separates the two. One thing separates them—when the refinement doubles, whether the length stays at 1 times or becomes 2 times. The exponent is continuous, and a coastline falls between 1 and 2, where a length means nothing unless the ruler used is stated. Fold less, so each step multiplies by 1.5, and the dimension falls: it is a quantity that can be designed continuously. To be said honestly—this paper does not prove that the limit curve exists; only finite-order polylines were computed. The 255.996094 and 2.001406 hold for the construction stated, and what does not move is only that the length ratio and the dimension are tied by one exponent. One last thing—when a quantity goes to infinity, the thing to doubt is the ruler and not the object. This curve’s length does diverge, but not because the curve is pathological—it is because a $D = 1$ ruler was applied to a $D = 2$ object. Change the ruler and the answer becomes finite again.

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