

When an Exact Formula Cannot Be Computed, Cutting It Short Brackets the Answer

— Inclusion--exclusion has $2^n - 1$ terms, and at $n = 60$ a billion terms a second takes **36.53** years — Yet stopping at an even term always overshoots and at an odd term always undershoots, with a gap shrinking as $1/(k+1)!$ — What makes it usable is not giving up but cutting — [Paper 360]

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— *From thinking to syncing.* —

Abstract

Inclusion--exclusion is an exact formula. This paper asks whether being exact makes it usable—the answer is that as it stands it is not, and cut short it is. No new mathematical theorem and no new law is claimed. First, the number of terms is $2^n - 1$, reaching **1073741823** at $n = 30$ (Section 2). Second, even at a billion terms a second, $n = 60$ takes **36.53** years (Section 2). Third, this is the core of the paper. Stopping at an even term always lies above the true value and at an odd term always below (Section 3). Fourth, the bracket shrinks as $1/(k+1)!$; the gap between $k = 5$ and $k = 6$ is $0.0013888889 = 1/6!$ (Section 3). Fifth, at $k = 12$ the residual against $1/e = 0.3678794412$ is 1.4980×10^{-10} (Section 3). Sixth, the separator is whether the terms alternate, not how many there are (Section 5).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed——inclusion–exclusion, the Bonferroni inequalities, and derangements are all standard. We do not build combinatorics——the general proof of the Bonferroni inequalities is not entered. It is named and no more. We do not build complexity theory——the hardness of the permanent and its approximation are not entered. They are named and no more. Implementation is not treated——floating-point rounding and big-number handling are not the subject. We do not build probability theory——error bounds for the Poisson approximation are not entered. We do not generalise——“an exact formula becomes usable when cut” is not claimed as a general law. It fails for formulae whose terms do not alternate. Relation to earlier papers: Paper 354 showed length ceasing to be the quantity that measures a curve——there the divergence was a mismatched ruler; here the explosion of terms is a mismatched method. Paper 353 showed an optimum lying in the middle rather than at an end——there too “more is better” was wrong; here stopping partway beats summing everything. Paper 300 showed whether two things share a root is decidable——by that test, “exact” and “computable” have distinct roots. Paper 357 showed one formula holding two optima——here one formula is both unusable and usable. What is added is converting the explosion of terms into time at six points, showing the alternating bracket at thirteen points, confirming that the gap equals $1/(k+1)!$ exactly, self-testing against an independently known value (9 derangements of 4), and putting the separator on whether the terms alternate.

1. Notation and the Question

Setting: Write the size of a union of n sets by inclusion–exclusion.

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{\emptyset \neq S \subseteq \{1, \dots, n\}} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|$$

The number of terms on the right is the number of non-empty subsets, that is $2^n - 1$.

Observation 1.1

This is not an approximation. It is an equality with no error term——mathematically a complete answer.

Observation 1.2

But the number of terms is exponential in n . Each increment of n doubles it——a complete answer of a length that cannot be written down.

Observation 1.3

Note that the signs alternate. The factor $(-1)^{|S|+1}$ —that alternation is the rescue in Section 3.

Question: Does being exact make it usable?

2. The Number of Terms Is $2^n - 1$

Calculation 2.1: Terms, and the time to sum them at a billion a second (computed in this paper).

n	Terms $2^n - 1$	Time
10	1023	0.0000 s
20	1048575	0.0010 s
30	1073741823	1.0737 s
40	1099511627775	0.31 hours
50	1125899906842623	13.03 days
60	1152921504606846975	36.53 years

Observation 2.2

Raising n by 10 multiplies the count by exactly 1024. Because $2^{10} = 1024$ —measured, 1024.000976: only the effect of the -1 survives.

Observation 2.3

The boundary lies between $n = 30$ and $n = 60$. From 1.0737 seconds to 36.53 years—thirty more sets turn something one can wait for into something one cannot.

Observation 2.4

$n = 60$ is not a large number. It just means sixty conditions—a commonplace size in practice, at which an exact formula stops reaching.

Observation 2.5

A faster machine does not solve it. A billion-fold speed-up buys 30 more in n —a linear improvement does not answer an exponential.

3. Cut It, and It Brackets

Calculation 3.1: The derangement probability truncated at k terms (computed in this paper; true value $1/e = 0.3678794412$).

k	Partial sum	Difference from truth	Side
0	1.0000000000	$+6.321 \times 10^{-1}$	above
1	0.0000000000	-3.679×10^{-1}	below
2	0.5000000000	$+1.321 \times 10^{-1}$	above
3	0.3333333333	-3.455×10^{-2}	below
4	0.3750000000	$+7.121 \times 10^{-3}$	above
5	0.3666666667	-1.213×10^{-3}	below
6	0.3680555556	$+1.761 \times 10^{-4}$	above
7	0.3678571429	-2.230×10^{-5}	below
8	0.3678819444	$+2.503 \times 10^{-6}$	above
12	0.3678794413	$+1.4980 \times 10^{-10}$	above

Observation 3.2

This is the core of the paper. Even always above, odd always below——without a single exception. So any two adjacent truncations give an interval containing the answer.

Observation 3.3

The bracket shrinks factorially. The gap between $k = 5$ and $k = 6$ is 0.0013888889 ——exactly $1/6! = 0.0013888889$.

Observation 3.4

Thirteen terms give nearly ten digits. The residual at $k = 12$ is 1.4980×10^{-10} ——not $2^{60} - 1$ terms but 13.

Observation 3.5

Here is the reversal. Summing everything takes 36.53 years; stopping partway gives ten digits at once——summing everything is not best. The same shape as the interior optimum of Paper 353.

Observation 3.6

It can be checked against an independently known value. Multiplying the partial sum at $k = 4$ by $4!$ gives 9——that there are 9 derangements of 4 objects is known independently. The formula closes correctly.

4. Why It Brackets

Observation 4.1

Because the terms alternate and their magnitudes decrease monotonically, $1/k!$ falls monotonically in k ——those two conditions alone give the alternating bracket.

Observation 4.2

Both conditions are needed. Without alternation there is no bracket; without decrease there is no convergence —drop either and the claim is false.

Observation 4.3

General inclusion--exclusion does not guarantee the second. The sizes of intersections need not decrease monotonically —so the Bonferroni inequalities say “it brackets” but not “the gap shrinks factorially”.

Observation 4.4

So the two claims have different strengths. “It brackets from both sides” holds generally —“ten digits in thirteen terms” is a property of the derangement example. They are not mixed.

Observation 4.5

To be stated honestly —this paper does not prove the Bonferroni inequalities. What is shown is that one example does bracket, and the size of its gap —the general proof is left to the literature.

5. The Separator: Do the Terms Alternate?

Observation 5.1

The separator —whether the terms alternate in sign, not how many there are.

	Signs	Truncated	Is the direction of error known
Inclusion--exclusion	alternating	brackets both sides	yes
A same-sign sum	identical	always from below	one side only

Observation 5.2

The rightmost column is the practical difference. Truncate a same-sign sum and all you know is “not yet enough”——truncate an alternating one and you know “it lies between these”.

Observation 5.3

Sheer number of terms is not the separator. Even 2^{60} terms give ten digits in thirteen if they alternate——having many is not the cause of the difficulty; having them all of one sign is.

Observation 5.4

The separator can be moved. A same-sign sum can be bracketed by supplying a separate sequence of upper bounds——but that sequence has to be brought in from outside. An alternating sum does it inside the formula.

Observation 5.5

To be stated honestly——36.53 years and 1.4980×10^{-10} rest on the assumption of a billion terms a second and on the derangement example. Another problem gives other numbers——what does not move is that the term count is $2^n - 1$ and that an alternating sum brackets.

Observation 5.6

One last thing——“exact” does not mean “usable”. And the operation that makes it usable is not usually the operation that abandons exactness——here, simply stopping the exact formula partway hands back the answer as a bracketed interval. What was given up is terms, not certainty.

6. The Census and the Close

1. $2^n - 1$ terms (Section 1): an equality, not an approximation.
2. 1073741823 terms at $n = 30$ (Calculation 2.1): 1.0737 seconds.
3. 36.53 years at $n = 60$ (Calculation 2.1): thirty more and it is unwaitable.

4. **+10 in n is 1024 times** (Observation 2.2): 2^{10} .
5. **Even above, odd below** (Observation 3.2): **not a single exception**.
6. **The $k = 5$ to $k = 6$ gap is $1/6!$** (Observation 3.3): **0.0013888889**.
7. **Residual 1.4980×10^{-10} at $k = 12$** (Calculation 3.1): **ten digits in 13 terms**.
8. **Times $4!$ gives 9** (Observation 3.6): **matches an independently known value**.
9. **Alternation and decrease are both needed** (Observation 4.2): **one alone will not do**.
10. **The count is not the separator** (Observation 5.3): **uniform sign is the cause**.

Closing: Inclusion--exclusion is not an approximation—it is an equality with no error term, mathematically a complete answer. But its term count $2^n - 1$ is exponential in n —even at a billion terms a second, $n = 30$ takes 1.0737 seconds while $n = 60$ takes **36.53** years. Thirty more sets turn something one can wait for into something one cannot. And $n = 60$ merely means sixty conditions, a commonplace size in practice. A billion-fold faster machine buys 30 more in n : a linear improvement does not answer an exponential. Yet cutting it short brackets the answer—on the derangement probability, stopping at an even term always lies above the true value $1/e = 0.3678794412$ and at an odd term always below—without a single exception. So any two adjacent truncations give an interval containing the answer. The bracket shrinks factorially: the gap between $k = 5$ and $k = 6$, 0.0013888889, is exactly $1/6!$. At $k = 12$ the residual is 1.4980×10^{-10} —ten digits from 13 terms, not $2^{60} - 1$. Here is the reversal—summing everything takes 36.53 years; stopping partway gives ten digits at once. Summing everything is not best. It brackets because the terms alternate and their magnitudes decrease monotonically, and both conditions are needed. General inclusion--exclusion does not guarantee the second, so “it brackets from both sides” holds generally while “ten digits in thirteen terms” is a property of this example. They are not mixed. One thing separates them—whether the terms alternate in sign, not how many there are. Truncate a same-sign sum and all you know is “not yet enough”; truncate an alternating one and you know “it lies between these”. Even 2^{60} terms give ten digits in thirteen if they alternate, so having many is not the cause of the difficulty—having them all of one sign is. To be said honestly—this paper does not prove the Bonferroni inequalities; what is shown is that one example does bracket, and the size of its gap. The 36.53 years and 1.4980×10^{-10} rest on stated assumptions. One last thing—“exact” does not mean “usable”. And the operation that makes it usable is not usually the operation that abandons exactness: here, simply stopping the exact formula partway hands back the answer as a bracketed interval. What was given up is terms, not certainty.

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