

Two Algebraically Equal Forms Return Answers Differing by 15 Digits

— The smaller root of a quadratic comes out with **0.59** correct digits from the naive form and **16** from the rationalised one — The cause is that $\sqrt{b^2 - 4c}$ agrees with b to **15.83** digits — Subtract two near-equals and exactly the agreeing digits vanish — [Paper 368]

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— *From thinking to syncing.* —

Abstract

It is tempting to think that mathematically identical formulae return identical answers. This paper asks whether that holds in finite precision—the answer is that it does not, and the gap reaches 15 digits. No new mathematical theorem and no new law is claimed. First, the smaller root of $x^2 + 10^8x + 1 = 0$ is computed by two equivalent formulae (Section 2). Second, this is the core of the paper. The naive form returns **0.59** correct digits and the rationalised one **16** (Section 2). Third, the cause is that $\sqrt{b^2 - 4c}$ agrees with b to **15.83** digits (Section 3). Fourth, for small b the two agree ($b = 3$ gives -0.3819660112501051 against -0.3819660112501052) (Section 3). Fifth, taking b from 10^2 to 10^{10} drops the naive form from 12.91 digits to 0 (Section 2). Sixth, the separator is whether the formula subtracts two near-equals, not its complexity or the size of the numbers (Section 5).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed——catastrophic cancellation, rationalisation, and floating-point arithmetic are all standard. We do not build numerical analysis——the general theory of rounding error and backward error analysis are not entered. They are named and no more. Implementation is not treated——arithmetic other than double precision, and compensated summation, are not the subject. Condition number is not treated——the conditioning of the problem itself is not the subject. Paper 330 treated the squaring of the condition number; what is treated here is the difference between two ways of writing one problem. We do not discuss root-finding——numerical methods for higher-degree equations are not the subject. We do not generalise——not every equivalent rewriting changes the digits. Only those in which cancellation occurs. Relation to earlier papers: Paper 330 showed the squaring of the condition number to be an identity rather than an approximation——there it was a property of the problem; here it is a property of how the problem is written. Paper 360 showed an exact formula that cannot be computed——there the trouble was the number of terms; here it is the subtraction of them. Paper 300 showed whether two things share a root is decidable——by that test, “mathematically equal” and “numerically equal” have distinct roots. Paper 355 showed an empirical law naming its own breakdown——here a formula names the b at which it breaks. What is added is giving the correct digits of both forms at five values, showing the gap reaching 15 digits, giving the causal 15.83-digit agreement as a number, confirming as a control that the two agree where no cancellation occurs, and putting the separator on whether two near-equals are subtracted.

1. Notation and the Question

Setting: Write the smaller root of $x^2 + bx + c = 0$ in two ways.

$$x_{\text{naive}} = \frac{-b + \sqrt{b^2 - 4c}}{2}, \quad x_{\text{rational}} = \frac{2c}{-b - \sqrt{b^2 - 4c}}$$

Take $b = 10^8$ and $c = 1$. The two are algebraically identical.

Observation 1.1

The two differ only by multiplying above and below by $(-b - \sqrt{b^2 - 4c})$. An algebraic rearrangement——mathematically there is no distinction.

Observation 1.2

But one contains a subtraction and the other does not. $-b + \sqrt{\cdots}$ against $-b - \sqrt{\cdots}$ ——a single sign apart.

Observation 1.3

This paper gives the consequence of that one sign as a number. Not “accuracy falls” but “how many digits”——knowing the size decides whether to rewrite.

Question: Do algebraically equal formulae return the same answer?

2. 0.59 Digits Against 16

Calculation 2.1: The two formulae at $b = 10^8$, $c = 1$ (computed in this paper, double precision).

Formula	Value
Naive	$-7.45058059692382812500 \times 10^{-9}$
Rationalised	$-1.00000000000000002092 \times 10^{-8}$
Relative error (naive)	2.549×10^{-1}
Correct digits (naive)	0.59
Correct digits (rationalised)	16

Observation 2.2

This is the core of the paper. The naive form has only **0.59** correct digits——even the first digit is doubtful. The relative error is 25 per cent.

Observation 2.3

The value -7.45×10^{-9} is recognisable. It is exactly half the double-precision spacing near 10^8 ($7.450580596923828 \times 10^{-9}$)——not a computed result but the residue of rounding, returned as the answer.

Calculation 2.4: Correct digits against b (computed in this paper).

b	Naive	Rationalised
10^2	12.91	16.00
10^4	8.95	16.00
10^6	5.12	16.00
10^8	0.59	16.00
10^{10}	0.00	16.00

Observation 2.5

Every two decades of b costs about 2 digits. $12.91 \rightarrow 8.95 \rightarrow 5.12 \rightarrow 0.59$ —regular. Not an accident.

Observation 2.6

At $b = 10^{10}$ it is 0. Not one digit is right—the formula returns an answer, and the answer carries no information.

3. The Cause Is a 15.83-Digit Agreement

Calculation 3.1: How closely $\sqrt{b^2 - 4c}$ agrees with b (computed in this paper).

b	Relative difference	Digits of agreement
10^8	1.490×10^{-16}	15.83
3	—	0.59

Observation 3.2

At $b = 10^8$ the two agree to 15.83 digits. With $c = 1$ and $b^2 = 10^{16}$, the term $4c$ sits 16 digits below b^2 —and that difference is buried under double-precision resolution.

Observation 3.3

Subtract two near-equals and exactly the agreeing digits vanish. Agreement to 15.83 digits leaves $16 - 15.83 = 0.17$ after the subtraction——consistent, as a count of digits, with the measured 0.59.

Observation 3.4

So only the subtraction is broken. The square root itself is computed correctly——which operation is at fault can be named.

Observation 3.5

The rationalised form turns the subtraction into an addition. $-b - \sqrt{\dots}$ adds two quantities of the same sign——no cancellation, so no digits are lost.

Observation 3.6

As a control, at $b = 3$ the two agree. -0.3819660112501051 against -0.3819660112501052 ——there $\sqrt{b^2 - 4c}$ agrees with b to only 0.59 digits, so subtracting destroys nothing.

4. The Formula Names Where It Breaks

Observation 4.1

The ratio of b^2 to $4c$ is the test. Double precision carries about 16 digits——the closer $b^2/(4c)$ gets to 10^{16} , the more dangerous.

Observation 4.2

So it is known before computing. Looking at b and c settles which formula to use——noticing after seeing the answer is too late.

Observation 4.3

The same shape as Paper 355. There an empirical law named the speed at which it broke——here a formula names the b at which it breaks. In both, the limit is written inside the formula.

Observation 4.4

The practical rule is simple. Use the two forms according to the sign——for $b > 0$ use the rationalised form for the smaller root; for $b < 0$ for the larger.

Observation 4.5

To be stated honestly——this paper performs no error analysis. What is shown is one example differing by 15 digits and that its cause is the agreement——general error bounds are left to the literature.

5. The Separator: Are Two Near-Equals Subtracted?

Observation 5.1

The separator——whether the formula subtracts two near-equal quantities; not its complexity and not the size of the numbers.

	Operation	At $b = 10^8$	At $b = 3$
Naive	$-b + \sqrt{\cdots}$ (cancels)	0.59 digits	16 digits
Rationalised	$-b - \sqrt{\cdots}$ (no cancellation)	16 digits	16 digits

Observation 5.2

The two right-hand columns do the deciding. At $b = 3$ there is no difference between them——only where cancellation occurs does the way of writing change the result.

Observation 5.3

Distinct roots by the criterion of Paper 300. “Mathematically equal” is fixed by algebra; “numerically equal” by the rounding of operations—the first can hold while the second fails.

Observation 5.4

The separator can be moved. Quadruple precision survives $b = 10^8$ —but the same happens at $b = 10^{16}$. Adding digits moves the boundary; it does not remove it.

Observation 5.5

To be stated honestly—0.59 digits and 15.83 digits are values under double precision with $b = 10^8$ and $c = 1$. Other conditions give other numbers—what does not move is the structure: as many digits are lost as agreed.

Observation 5.6

One last thing—“tidying up a formula” does not necessarily leave the answer unchanged. Algebraically it changes nothing; numerically it changes 15 digits—that a tidied formula reads well and that it returns the right answer are different properties.

6. The Census and the Close

1. The two forms are algebraically identical (Section 1): merely rationalised.
2. The naive form gives 0.59 digits (Calculation 2.1): 25 per cent relative error.
3. The rationalised form gives 16 (Calculation 2.1): a gap of 15.
4. About 2 digits per two decades of b (Observation 2.5): regular.
5. 0 digits at $b = 10^{10}$ (Calculation 2.4): the answer carries no information.
6. $\sqrt{b^2 - 4c}$ agrees with b to 15.83 digits (Calculation 3.1): the cause.
7. $16 - 15.83 = 0.17$ (Observation 3.3): the count of digits is consistent.
8. Only the subtraction is broken (Observation 3.4): the square root is right.

9. At $b = 3$ the two agree (Observation 3.6): the control.

10. More precision only moves the boundary (Observation 5.4): it cannot be removed.

Closing: It is tempting to think that mathematically identical formulae return identical answers—in finite precision they do not. Write the smaller root of $x^2 + 10^8x + 1 = 0$ two ways—the naive $(-b + \sqrt{b^2 - 4c})/2$ and the rationalised $2c/(-b - \sqrt{b^2 - 4c})$ are algebraically identical and differ by a single sign. Yet the naive form returns **0.59** correct digits and the rationalised one **16**. The naive form’s relative error is 25 per cent, so even its first digit is doubtful. The value it returns, -7.45×10^{-9} , is exactly half the double-precision spacing near 10^8 —not a computed result but the residue of rounding. The cause can be named—at $b = 10^8$, $\sqrt{b^2 - 4c}$ agrees with b to **15.83** digits (with $c = 1$ and $b^2 = 10^{16}$, the term $4c$ is buried under double-precision resolution). Subtract two near-equals and exactly the agreeing digits vanish: $16 - 15.83 = 0.17$, consistent as a count with the measured 0.59. Only the subtraction is broken; the square root itself is computed correctly. The rationalised form adds two quantities of the same sign, so nothing cancels. And the formula names where it breaks—taking b from 10^2 to 10^{10} drops the naive form $12.91 \rightarrow 8.95 \rightarrow 5.12 \rightarrow 0.59 \rightarrow 0$, about 2 digits per two decades, regularly. Since the danger grows as $b^2/(4c)$ approaches the 16 digits of double precision, which formula to use is settled before computing. The same shape as the empirical law naming its own breaking speed in Paper 355. A control shows this is a matter of how it is written—at $b = 3$ the two agree (-0.3819660112501051 against -0.3819660112501052), because there $\sqrt{b^2 - 4c}$ agrees with b to only 0.59 digits and subtracting destroys nothing. One thing separates them—whether the formula subtracts two near-equal quantities; not its complexity and not the size of the numbers. Quadruple precision survives $b = 10^8$, but the same happens at $b = 10^{16}$: adding digits moves the boundary, it does not remove it. To be said honestly—this paper performs no error analysis; it shows one example differing by 15 digits and that its cause is the agreement. What does not move is the structure: as many digits are lost as agreed. One last thing—“tidying up a formula” does not necessarily leave the answer unchanged. Algebraically it changes nothing; numerically it changes 15 digits. That a tidied formula reads well and that it returns the right answer are different properties.

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