

Put a Number on a Pendulum’s “Small Angle” and Ten Degrees Costs 164.78 Seconds a Day

— two degrees costs **6.58** s/day and thirty degrees **1504.12** s/day (**25** minutes) — yet at **170** degrees the period is only **2.4394** times, and it reaches three times at **175.8765** degrees — the divergence bites only in the last fraction of a degree — [Paper 396]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026

— *From thinking to syncing.* —

Abstract

We are taught that a pendulum is isochronous if the amplitude is small. This paper asks how small is small—even ten degrees costs **164.78** seconds a day, which is useless for a clock. No new theorem or law is claimed. First, the period ratio is $T/T_0 = (2/\pi)K(\sin^2(\theta/2))$, a complete elliptic integral of the first kind (Section 1). Second, one degree costs **1.64** s/day and two degrees **6.58** s/day (Section 2). Third, and this is the core of the paper. Ten degrees costs **164.78** s/day: two minutes and forty-five seconds (Section 2). Fourth, thirty degrees costs **1504.12** s/day (**25.07** minutes) and ninety degrees **4.33** hours (Section 2). Fifth, yet at **170** degrees the period is only **2.4394** times, and reaches three times at **175.8765** degrees (Section 3). Sixth, the separator is whether the restoring force goes as the displacement or as its sine (Section 5).

Scope of this paper (scope note): No new theorem or law is claimed—the exact period of a simple pendulum, its elliptic-integral form and Huygens’s cycloidal pendulum are all known. No clock is designed—escapements and temperature compensation are not the subject. Only geometry and mechanics. Damping is not treated—air resistance and bearing friction are not the subject. Only a conservative system. No measurement is claimed—the accuracy of real pendulum clocks is outside this paper. Only what the formula gives. Rigid bodies are not treated—only a point-mass simple pendulum; the moment of inertia of a physical pendulum is not the subject. Driving is not treated—how the amplitude is sustained is not the subject. Relation to earlier papers: Paper 379 showed that “twice as hot” has three answers—here the phrase “small angle” contains no number. Paper 391 showed that the significant wave height is neither the mean nor the maximum—in both, a word hides a number. Paper 392 showed that the boiling point is a property of substance and pressure together—here the period is a property of length and amplitude together. Paper 300 showed that same-root and different-root can be decided—the harmonic oscillator and the simple pendulum are different-root by that criterion. What is added is putting numbers on the period ratio and daily error at seven amplitudes, obtaining 164.78 s/day at ten degrees, showing that 170 degrees does not reach three times, locating the tripling amplitude at 175.8765 degrees, and placing the separator at “what the restoring force is proportional to”.

1. Notation and the Question

Setting: a simple pendulum of length ℓ obeys $\ddot{\theta} = -(g/\ell) \sin \theta$. Dividing the period at amplitude θ by the small-amplitude period $T_0 = 2\pi\sqrt{\ell/g}$ gives

$$\frac{T}{T_0} = \frac{2}{\pi} K\left(\sin^2 \frac{\theta}{2}\right)$$

where K is the complete elliptic integral of the first kind, computed below by the arithmetic--geometric mean.

Observation 1.1

isochronism comes from the approximation $\sin \theta \approx \theta$. The equation itself is not isochronous—drop the approximation and the period depends on the amplitude.

Observation 1.2

the word “small” contains no number. How small is small—it is fixed by the accuracy demanded.

Observation 1.3

for a clock, accuracy is measured in seconds per day. $(T/T_0 - 1) \times 86400$ —that quantity defines “small” here.

Question: how many degrees is a pendulum’s “small angle”?

2. Ten Degrees Costs 164.78 Seconds a Day

Computation 2.1: period ratio and daily error against amplitude (computed in this paper).

Amplitude [deg]	T/T_0	Daily error [s]
1	1.000019039	1.64
2	1.000076160	6.58
5	1.000476172	41.14
10	1.001907188	164.78
30	1.017408798	1504.12
90	1.180340599	15581.43
170	2.439362720	124360.94

Observation 2.2

this is the core of the paper. Ten degrees looks small enough by any intuition—and still costs 164.78 seconds a day, two minutes and forty-five seconds.

Observation 2.3

only at two degrees does it fall to 6.58 s/day. Which is why working pendulum clocks hold the amplitude to a few degrees—the meaning of “small” is back-computed from the accuracy required.

Observation 2.4

the error grows as the square of the amplitude. From 1.64 s at one degree to 6.58 s at two, very nearly four times—the $1 + \theta^2/16$ approximation governs this range.

Observation 2.5

thirty degrees costs 1504.12 s (25.07 minutes). Twenty-five minutes in a day—no longer a clock in any sense.

Observation 2.6

ninety degrees costs 15581.43 s (4.33 hours). Released from the horizontal—and even so the period is only 1.18 times.

3. At 170 Degrees It Still Does Not Reach Three Times

Computation 3.1: large amplitudes (computed in this paper).

Amplitude [deg]	T/T_0
90	1.180341
170	2.439363
175	2.877664
175.8765	3.000000
179	3.901065
179.9	5.366867

Observation 3.2

even at 170 degrees it is only 2.4394 times. Almost lifted to the vertical——three times arrives only at 175.8765 degrees.

Observation 3.3

T/T_0 diverges logarithmically as $\theta \rightarrow 180^\circ$. At 179.9 degrees it is still 5.3669——the divergence is slow and bites only in the last fraction of a degree.

Observation 3.4

so “the period diverges at large amplitude” does not act in any practical range. A tenth of a degree short of vertical gives 5.37——a qualitative statement distorting a quantitative estimate.

Observation 3.5

the two ends have different causes. θ^2 on the small side, $\ln(1/(180^\circ - \theta))$ on the large——two asymptotic forms of one curve.

Observation 3.6

T/T_0 is monotone increasing in amplitude. It never falls from 1 degree to 174——no peak and no valley.

4. Where the Small-Angle Approximation Actually Breaks

Observation 4.1

the small-angle approximation is $T/T_0 \approx 1 + \theta^2/16$. With θ in radians——its error at ten degrees is 3.329×10^{-6} .

Observation 4.2

at ninety degrees the error is 2.613×10^{-2} .About 7800 times the value at ten degrees—the approximation fails faster than quadratically.

Observation 4.3

at ten degrees the approximation is perfectly good.An error of 3.3×10^{-6} is 0.29 seconds a day—the approximation is not the problem; the true value itself is off by 164.78 seconds.

Observation 4.4

this is easily conflated.“The approximation is bad” and “it is not isochronous” are different—at ten degrees the problem is the second.

Observation 4.5

they are different-root by the criterion of Paper 300.The approximation error is set by mathematics, the failure of isochronism by mechanics—one word, “error”, with two roots.

5. Separator: Displacement or Its Sine

Observation 5.1

separator—is the restoring force proportional to θ or to $\sin \theta$.

System	Restoring force	Isochronism
Harmonic oscillator (spring)	proportional to displacement	exactly isochronous
Simple pendulum	proportional to $\sin \theta$	not isochronous
Cycloidal pendulum	proportional to arc length	exactly isochronous at all amplitudes

Observation 5.2

the third row solves the problem. Shown by Huygens in 1673——make the path a cycloid and the period does not change however large the amplitude.

Observation 5.3

so isochronism is a question about the shape of the path, not about the law of force. Under the same gravity——a circular arc is not isochronous and a cycloid is.

Observation 5.4

and yet working clocks did not use the cycloid. The friction of the cheeks exceeded the gain in isochronism——a correct solution can lose in implementation.

Observation 5.5

to be honest——both 164.78 s/day and 175.8765 degrees are values for an undamped conservative system. A real pendulum loses amplitude, so its error changes with time——what does not move is that ten degrees is not “small” for a clock, and that 170 degrees does not reach three times.

Observation 5.6

one last thing——a condition that says “if it is small” is not a condition until a number is written. Ten degrees looks small and is not——until the required accuracy is fixed, “small” is undetermined.

6. Roll Call and Closing

1. $T/T_0 = (2/\pi)K(\sin^2(\theta/2))$ (Section 1): an elliptic integral.
2. 1.64 s/day at one degree (Computation 2.1): only here is it usable.
3. 6.58 s/day at two degrees (Computation 2.1): the range of working clocks.
4. 164.78 s/day at ten degrees (Computation 2.1): two minutes forty-five.

5. **25.07 minutes a day at thirty degrees** (Observation 2.5): **no longer a clock.**
6. **4.33 hours a day at ninety** (Observation 2.6): **the ratio is still 1.18.**
7. **2.4394 at 170 degrees** (Computation 3.1): **short of three.**
8. **Three times at 175.8765 degrees** (Computation 3.1): **the divergence is last-minute.**
9. **Approximation error 3.3×10^{-6} at ten degrees** (Observation 4.1): **the approximation is fine.**
10. **A cycloid is isochronous at all amplitudes** (Observation 5.2): **a question of path shape.**

Closing: we are taught that a pendulum is isochronous if the amplitude is small—but the word “small” contains no number. Exactly, $T/T_0 = (2/\pi)K(\sin^2(\theta/2))$ —isochronism comes from the approximation $\sin \theta \approx \theta$, and the equation itself is not isochronous. How small counts as small is fixed by the accuracy demanded. Ten degrees costs **164.78** seconds a day—two minutes and forty-five seconds, and ten degrees looks small enough by any intuition while being useless for a clock. Only at two degrees does it fall to 6.58 s/day, which is why working pendulum clocks hold the amplitude to a few degrees—the meaning of “small” is back-computed from the accuracy required. The error grows as the square of the amplitude (1.64 s at one degree, 6.58 s at two), reaching 1504.12 s (25.07 minutes) at thirty degrees and 4.33 hours at ninety. At the other end, the divergence bites far later than expected—at 170 degrees the period is only **2.4394** times, and three times arrives at **175.8765** degrees. T/T_0 diverges logarithmically at 180 degrees, but at 179.9 degrees it is still 5.3669. “The period diverges at large amplitude” is a qualitative statement that distorts the quantitative estimate. The two ends have different causes: θ^2 on the small side and $\ln(1/(180^\circ - \theta))$ on the large, two asymptotic forms of one curve. The small-angle approximation is not what breaks—the error of $T/T_0 \approx 1 + \theta^2/16$ at ten degrees is 3.329×10^{-6} , which is 0.29 seconds a day. At ninety degrees it opens to 2.613×10^{-2} , about 7800 times larger, but at ten degrees the problem is not the approximation error—it is that the true value itself is off by 164.78 seconds. “The approximation is bad” and “it is not isochronous” are different, and different-root by the criterion of Paper 300: mathematics on one side, mechanics on the other. One thing does the separating—is the restoring force proportional to θ or to $\sin \theta$. A spring is exactly isochronous, a simple pendulum is not, and a cycloidal pendulum is exactly isochronous at every amplitude (Huygens, 1673). So isochronism is a question about the shape of the path, not the law of force: under the same gravity a circular arc fails and a cycloid succeeds. And yet working clocks did not use the cycloid, because the friction of the cheeks exceeded the gain—a correct solution can lose in implementation. To be honest—both 164.78 s/day and 175.8765 degrees are values for an undamped conservative system, and a real pendulum loses amplitude so its error changes with time. What does not move is that ten degrees is not “small” for a clock, and that 170 degrees does not reach three times. One last thing—a condition that says “if it is small” is not a condition until a number is written. Ten degrees looks small and is not. Until the required accuracy is fixed, “small” is undetermined.

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Yuuki YAMAGISHI (山岸 勇輝)

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