

The Order of Temperature Is Not the Order of Hotness

— in a two-level system $T = -2.466303$ is hotter than $T = +2.466303$, and $T = -0.455120$ is the hottest of the five — what orders them correctly is not T but $\beta = 1/T$ — as a quantity for comparison, temperature is broken — [Paper 403]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

September 2026

— *From thinking to syncing.* —

Abstract

Higher temperature sounds like hotter. This paper shows that the numerical order of temperature and the order of hotness do not agree—a negative temperature is hotter than every positive one. No new theorem or law is claimed. First, in a two-level system $1/T = (k/\epsilon) \ln((1-f)/f)$, and the temperature turns negative once the excited fraction f passes one half (Section 1). Second, at $f = 0.5$, β is exactly 0 and the temperature diverges (Section 2). Third, and this is the core of the paper. By temperature the order is 0.6, 0.9, 0.1, 0.4, 0.5; by hotness it is 0.9, 0.6, 0.5, 0.4, 0.1—they disagree (Section 2). Fourth, $T = -2.466303$ and $T = +2.466303$ differ only in sign, and the former is the hotter (Section 2). Fifth, ordering by $\beta = 1/T$ is correct, and negative temperatures are hotter than all positive ones (Section 3). Sixth, the separator is whether the energy is bounded above (Section 5).

Scope of this paper (scope note): No new theorem or law is claimed——negative absolute temperature, the statistics of a two-level system and ordering by $\beta = 1/(kT)$ are all known in statistical mechanics.No thermodynamics is built——formulations of the second law and efficiencies at negative temperature are not entered. Only the ordering inside one model.No experiment is claimed——realisations in nuclear spin systems and cold atoms are named, not examined. The numbers are the model's.Equilibrium is assumed——negative-temperature states are metastable and decay by relaxation to the lattice; that timescale is not the subject.The energy origin is fixed——ground at 0 and excited at ε ; shifting the origin does not change the ordering.Units are not treated——temperature is written in units of ε/k .Relation to earlier papers: Paper 379 showed that “twice as hot” has three answers——here the ordering of temperature itself breaks.Paper 391 showed that a representative value does not represent——here temperature does not represent the ordering.Paper 394 showed that the same flow is realised in two states——here the same absolute value of temperature carries two hotnesses.Paper 300 showed that same-root and different-root can be decided——“higher temperature” and “hotter” are different-root by that criterion.What is added is putting numbers on temperature and β at five excitation fractions, setting the two orderings side by side to show they disagree, showing that β is exactly zero at $f = 0.5$, confirming with β that negative temperatures exceed all positive ones, and placing the separator at “bounded above or not”.

1. Notation and the Question

Setting: of N two-level spins let n be excited, with $f = n/N$. The entropy is $S = k \ln \binom{N}{n}$ and the energy $E = n\varepsilon$.

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{k}{\varepsilon} \ln \frac{1-f}{f}, \quad \beta \equiv \frac{1}{kT} = \frac{1}{\varepsilon} \ln \frac{1-f}{f}$$

Below, temperature is written in units of ε/k .

Observation 1.1

for $f < 1/2$ the logarithm's argument exceeds one and $T > 0$.The ordinary case——for $f > 1/2$ the logarithm is negative and $T < 0$.

Observation 1.2

at $f = 1/2$ the logarithm is exactly zero. $\beta = 0, T = \pm\infty$ ——a point of divergent temperature sits in the middle of the range.

Observation 1.3

negative temperature exists because the energy is bounded above. $E \leq N\epsilon$ —without a ceiling, S increases monotonically and $\partial S/\partial E$ stays positive.

Question: is being at a higher temperature the same as being hotter?

2. The Two Orderings Disagree

Computation 2.1: temperature and β against excitation fraction (computed in this paper).

Excited fraction f	Temperature T [ϵ/k]	$\beta = 1/T$
0.1	+0.455120	+2.197225
0.4	+2.466303	+0.405465
0.5	∞	0.000000
0.6	−2.466303	−0.405465
0.9	−0.455120	−2.197225

Observation 2.2

this is the core of the paper. By increasing temperature the order is 0.6, 0.9, 0.1, 0.4, 0.5—by increasing hotness (decreasing β) it is 0.9, 0.6, 0.5, 0.4, 0.1. The two sequences disagree.

Observation 2.3

the temperatures at $f = 0.4$ and $f = 0.6$ differ only in sign. +2.466303 and −2.466303—equal in magnitude, opposite in hotness.

Observation 2.4

at $f = 0.5$, β is exactly $0.000000.\ln(0.5/0.5) = 0$ —the temperature diverges while β merely passes through an ordinary point.

Observation 2.5

the hottest is $f = 0.9$, at $T = -0.455120$. Second lowest of the five by numerical temperature—an example of the two orders pointing opposite ways, inside the table.

Observation 2.6

the largest β on the negative side is below the smallest on the positive side. $-0.405465 < +0.405465$ —a negative-temperature system is hotter than every positive-temperature one.

3. What Reorders Them Is β

Observation 3.1

β decreases monotonically in f . From $+2.197225$ to -2.197225 —without a jump.

Observation 3.2

T jumps from $+\infty$ to $-\infty$ at $f = 0.5$. Discontinuous—a continuous change of the physical state mapped discontinuously by T .

Observation 3.3

the direction of heat flow is set by β . From smaller β to larger—not by the ordering of T .

Observation 3.4

hence the sequence “ -0 hottest, $+0$ coldest”. $+0 \rightarrow +\infty \rightarrow -\infty \rightarrow -0$ ——the temperature scale is not a line; the line is on the β side.

Observation 3.5

the entropy is maximal at $f = 0.5$. $S(0.5) > S(0.4)$ and $S(0.5) > S(0.6)$ ——past it, $\partial S/\partial E$ turns negative and so does the temperature.

Observation 3.6

the shape is that of Paper 391. There a representative value did not represent——here temperature does not represent the ordering. In both, a familiar single number fails at its job.

4. The Word “Temperature” Names Two Things

Observation 4.1

one is the definition, the reciprocal of $\partial S/\partial E$. A formal quantity——which may carry a sign.

Observation 4.2

the other is “which way heat flows on contact”. An operational quantity——ordered by β .

Observation 4.3

in ordinary systems the two agree. Within $T > 0$ the orderings of T and β are reciprocal, so the order is preserved——which is why no distinction is needed.

Observation 4.4

the moment a negative temperature appears, they separate. The order by T and by β differ—they agreed only because the range was restricted.

Observation 4.5

they are different-root by the criterion of Paper 300. The definition is set by a derivative, hotness by contact between two systems—one word with two roots.

5. Separator: Is the Energy Bounded Above

Observation 5.1

separator—does the energy have a ceiling or not.

System	Energy	Entropy	Negative temperature
Two-level spins	bounded above	maximal partway	exists
Translation (a gas)	unbounded	monotone increasing	does not exist

Observation 5.2

in the second row $\partial S/\partial E$ is always positive. T stays positive—a gas cannot be brought to a negative temperature.

Observation 5.3

so negative temperature is a property of the system, not a result of cooling. No amount of cooling produces it—a bounded spectrum is the condition.

Observation 5.4

the separator can be moved. Couple the spins to the lattice and they join an unbounded degree of freedom, and the negative temperature collapses—which is why it is only metastable.

Observation 5.5

to be honest—both ± 2.466303 and -0.455120 are values of a two-level model. With three or more levels the numbers change—what does not move is that β is monotone in f and that T jumps at $f = 0.5$.

Observation 5.6

one last thing—“negative temperature is colder than absolute zero” is wrong twice. Far from colder, it is hotter than every positive temperature—and it lies not below absolute zero but on the far side of infinity.

6. Roll Call and Closing

1. $\beta = (1/\varepsilon) \ln((1 - f)/f)$ (Section 1): **negative for $f > 1/2$.**
2. $\beta = 0.000000$ at $f = 0.5$ (Computation 2.1): **T diverges.**
3. By temperature: **0.6, 0.9, 0.1, 0.4, 0.5** (Observation 2.2): by hotness: **0.9, 0.6, 0.5, 0.4, 0.1.**
4. ± 2.466303 (Observation 2.3): **equal magnitude, opposite hotness.**
5. The hottest is $T = -0.455120$ (Observation 2.5): **second lowest numerically.**
6. Negative beats every positive (Observation 2.6): **compare with β .**
7. β monotone, T jumps (Observations 3.1, 3.2): **a problem of the map.**
8. S maximal at $f = 0.5$ (Observation 3.5): **past it, the sign turns.**
9. A gas has no negative temperature (Observation 5.2): **no ceiling.**
10. Not below absolute zero (Observation 5.6): **beyond infinity.**

Closing: higher temperature sounds like hotter—but the numerical order of temperature and the order of hotness do not agree. In a two-level system $\beta = 1/(kT) = (1/\varepsilon) \ln((1 - f)/f)$, and the temperature turns negative once f passes one half—at $f = 1/2$ the logarithm is exactly zero, $\beta = 0.000000$ and $T = \pm\infty$. Negative temperature exists because the energy is bounded above at $N\varepsilon$; without a ceiling S increases monotonically and $\partial S/\partial E$ stays positive. The two orderings disagree—by increasing temperature: **0.6, 0.9, 0.1, 0.4, 0.5**; by increasing hotness:

0.9, 0.6, 0.5, 0.4, 0.1. The temperatures at $f = 0.4$ and $f = 0.6$ are $+2.466303$ and -2.466303 —equal in magnitude, opposite in hotness. The hottest is $f = 0.9$ at $T = -0.455120$, which is only the second lowest of the five by numerical temperature. And the largest β on the negative side, -0.405465 , is below the smallest on the positive side, $+0.405465$: a negative-temperature system is hotter than every positive-temperature one. What reorders them is β — β decreases monotonically from $+2.197225$ to -2.197225 without a jump, while T jumps from $+\infty$ to $-\infty$ at $f = 0.5$. A continuous change of the physical state, mapped discontinuously by T . Heat flows from smaller β to larger, not by the ordering of T , giving the sequence $+0 \rightarrow +\infty \rightarrow -\infty \rightarrow -0$: the temperature scale is not a line, and the line is on the β side. The entropy is maximal at $f = 0.5$, and past it $\partial S/\partial E$ turns negative. The word “temperature” names two things—the reciprocal of $\partial S/\partial E$, a formal quantity that may carry a sign; and “which way heat flows on contact”, an operational quantity ordered by β . Within $T > 0$ the two orderings are reciprocal and agree, which is why no distinction is needed—and they separate the moment a negative temperature appears. They agreed only because the range was restricted. By the criterion of Paper 300 they are different-root: a derivative on one side, contact between two systems on the other, the shape of Paper 391. One thing does the separating—does the energy have a ceiling or not. Two-level spins are bounded above, their entropy is maximal partway, and negative temperature exists; translation in a gas is unbounded, its entropy increases monotonically, $\partial S/\partial E$ is always positive, and it does not. So negative temperature is a property of the system and not a result of cooling: no amount of cooling produces it, and a bounded spectrum is the condition. Couple the spins to the lattice and they join an unbounded degree of freedom, and the state collapses—which is why it is only metastable. To be honest—both ± 2.466303 and -0.455120 are values of a two-level model, and with more levels the numbers change. What does not move is that β is monotone in f and that T jumps at $f = 0.5$. One last thing—“negative temperature is colder than absolute zero” is wrong twice. Far from colder, it is hotter than every positive temperature. And it lies not below absolute zero but on the far side of infinity.

References

1. Purcell, E. M., Pound, R. V. (1951). A Nuclear Spin System at Negative Temperature. *Physical Review*.
2. Ramsey, N. F. (1956). Thermodynamics and Statistical Mechanics at Negative Absolute Temperatures. *Physical Review*.
3. Braun, S., Ronzheimer, J. P., Schreiber, M., et al. (2013). Negative Absolute Temperature for Motional Degrees of Freedom. *Science*.
4. Kittel, C., Kroemer, H. (1980). *Thermal Physics* (2nd ed.). New York: W. H. Freeman.
5. Yamagishi, Y. (2026). *The Three Hundredth Census: Whether Two Things Share a Root Is Decidable* [Paper 300]. Zenodo. 10.5281/zenodo.22171914
6. Yamagishi, Y. (2026). “Twice as Hot” Has Three Answers, and Only One Gives a Ratio of **2** [Paper 379]. Zenodo. 10.5281/zenodo.22698802
7. Yamagishi, Y. (2026). *The Significant Wave Height Is Neither the Mean Nor the Maximum* [Paper 391]. Zenodo. 10.5281/zenodo.22699212

8. Yamagishi, Y. (2026). *The Same Flow Is Two Different States: Measuring the Flow Does Not Tell You Which One You Are In* [Paper 394]. Zenodo. 10.5281/zenodo.22699272

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)