

A Numerical Coincidence Becomes Evidence of a Common Root Only When There Is a Bijection

— parenthesis strings, binary trees, triangulations and lattice paths agree exactly from $n = 0$ to 8 (1, 1, 2, 5, 14, 42, 132, 429, 1430) — yet the central binomial coefficient also agrees at $n = 0$ and parts company at $n = 1$ — the number of agreeing terms is not the strength of the evidence — [Paper 405]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026

— *From thinking to syncing.* —

Abstract

Paper 300 wrote that numerical agreement alone is not taken as a common root. This paper asks what then does establish one—the answer is a bijection, not a count of agreeing terms. No new theorem or law is claimed. First, four independent counts agree exactly from $n = 0$ to 8 (1, 1, 2, 5, 14, 42, 132, 429, 1430) (Section 2). Second, $C_{10} = 16796$, and $\binom{2n}{n}/(n+1)$ divides exactly for every n from 0 to 19 (Section 2). Third, and this is the core of the paper. The central binomial coefficient also agrees at $n = 0$ and parts company at $n = 1, 2$ against 1. The number of agreeing terms is not the strength of the evidence (Section 3). Fourth, the four share a root because a bijection can be constructed (Section 4). Fifth, the asymptotic $4^n/(n^{3/2}\sqrt{\pi})$ is off by 11.38 per cent at $n = 10$ and 0.11 per cent at $n = 1000$ (Section 3). Sixth, the separator is “agreement on finitely many values” against “a constructed correspondence” (Section 5).

Scope of this paper (scope note): No new theorem or law is claimed—the Catalan numbers, the agreement of these four counts, bijective proofs and the asymptotic form are all known in combinatorics. No bijection is constructed—the explicit correspondences (rotation maps, depth-first encodings) are known; this paper uses only their existence. No combinatorics is built—generating functions and the general theory of lattice paths are not entered. Four sequences are counted and compared. Other coincidences are not treated—agreements among Fibonacci or Bell numbers are not the subject. Probability is not treated—how likely a chance agreement is is not the subject. Only that the number of agreements is not evidence. The definition of “common root” is unchanged—the criterion of Paper 300 is used as it stands. Relation to earlier papers: Paper 300 showed that same-root and different-root can be decided—this paper supplies one example on the “same root” side. Paper 400 showed that a numerical coincidence can look like meaning—here, by contrast, is a genuine common root. Paper 388 showed two different quantities converging to one constant—that was agreement in the limit, and not a common root. Paper 398 showed one operation becoming different quantities by position—here different operations become one quantity. What is added is implementing the four counts independently and confirming agreement from $n = 0$ to 8, showing that the central binomial coefficient agrees at $n = 0$ and diverges at $n = 1$, confirming the recurrence as an independent route, giving the asymptotic error at $n = 10$ and $n = 1000$, and placing the separator at “finitely many agreements or a constructed correspondence”.

1. Notation and the Question

Setting: let $C_n = \binom{2n}{n}/(n+1)$ and count the following four independently. (a) balanced parenthesis strings of length $2n$. (b) binary trees on n nodes. (c) triangulations of a convex $(n+2)$ -gon. (d) $n \times n$ lattice paths not crossing the diagonal.

Observation 1.1

the four are entirely different objects. Strings, trees, polygons, paths—nothing in the definitions says why the counts should agree.

Observation 1.2

Paper 300 wrote that numerical agreement alone is not a common root. There is a criterion—so: do these four meet it?

Observation 1.3

adding more agreeing terms does not produce evidence. Finitely many can always be arranged——what is needed is a different kind of evidence.

Question: when may agreement of numbers be called a common root?

2. Four Counts Agree Exactly from $n = 0$ to 8

Computation 2.1: the four counted independently (computed in this paper).

n	C_n	(a) strings	(b) trees	(c) triangulations	(d) paths
0	1	1	1	1	1
1	1	1	1	1	1
2	2	2	2	2	2
3	5	5	5	5	5
4	14	14	14	14	14
5	42	42	42	42	42
6	132	132	132	132	132
7	429	429	429	429	429
8	1430	1430	1430	1430	1430

Observation 2.2

all four agree in all nine rows. Separately implemented counts——and they agree with $\binom{2n}{n}/(n+1)$ as well.

Observation 2.3

$C_{10} = 16796$. An independently known value——neither the formula nor the implementations are off.

Observation 2.4

$\binom{2n}{n}$ is always divisible by $n + 1$. Confirmed for $n = 0$ to 19—that it is an integer at all is not an accident.

Observation 2.5

the recurrence $C_{n+1} = \sum_i C_i C_{n-i}$ also holds for $n = 0$ to 12. An independent route—closed form and recurrence return the same numbers.

Observation 2.6

that is nine agreeing terms and two independent formulae. It looks sufficient—and it is still not evidence of a common root.

3. The Central Binomial Coefficient Also Agrees at $n = 0$

Computation 3.1: a control for false agreement (computed in this paper).

n	C_n	$\binom{2n}{n}$
0	1	1 (agrees)
1	1	2 (parts)
2	2	6
3	5	20
10	16796	184756

Observation 3.2

this is the core of the paper. They agree at $n = 0$ and part at $n = 1$ —whether one term agrees, or two, or nine, that is not evidence of a common root. It is only a count.

Observation 3.3

at $n = 10$ the gap is exactly a factor of 11.16796 and 184756—look at the first term alone and they are the same sequence.

Observation 3.4

the asymptotic form is another agreement that is not one. $4^n / (n^{3/2} \sqrt{\pi})$ —off by 11.38 per cent at $n = 10$, 1.13 at $n = 100$ and 0.11 at $n = 1000$.

Observation 3.5

the error shrinks with n and never becomes zero. The ratio only approaches one—approaching and being equal are different.

Observation 3.6

the shape is that of Paper 388. There two different quantities went to the same $1/e$ —a shared limit is not a shared root, and approaching is not evidence.

4. The Evidence of a Common Root Is the Correspondence Itself

Observation 4.1

the four share a root because a bijection can be constructed. Between strings and trees, between strings and paths, and so on—a rule mapping each object to one object.

Observation 4.2

a bijection is stronger than agreement of numbers. It contains equality of counts and adds “which corresponds to which”—the numerical agreement is a corollary of it.

Observation 4.3

so a bijection gives agreement at every n . Not a check of finitely many—and the number of checks stops mattering.

Observation 4.4

the central binomial coefficient admits no such correspondence. The agreement at $n = 0$ is only that both happen to be 1—try to build a map and it fails at $n = 1$.

Observation 4.5

this is precisely what the criterion of Paper 300 says. Numerical agreement alone is not a common root—this paper sets an example that meets it and one that does not on the same page.

Observation 4.6

the criterion has the same shape in physics. Getting the same equation is not a common root; the same variables must play the same parts—a separator this corpus has used repeatedly.

5. Separator: Finitely Many Agreements, or a Constructed Correspondence

Observation 5.1

separator—is the agreement over finitely many values, or is there a correspondence between the objects.

Relation	Agreement	Bijection	Common root
Strings and trees	every n	exists	yes
C_n and $\binom{2n}{n}$	only $n = 0$	none	no
C_n and the asymptotic	nowhere	none	no

Observation 5.2

the third row agrees nowhere and resembles most closely. 0.11 per cent at $n = 1000$ —resembling and sharing a root are entirely different.

Observation 5.3

the second row agrees once and is wholly unrelated. Parting at $n = 1$ —counting agreements gives no ordering.

Observation 5.4

the third column alone decides. Is there a bijection—no numbers need be looked at.

Observation 5.5

to be honest—this paper constructs no bijection; it uses the existence of known ones. The constructions are in the textbooks—what does not move is that agreement at $n = 0$ is not evidence and that they part at $n = 1$.

Observation 5.6

one last thing—“nine terms agreed” is a quantity of evidence, not a kind. Ten or a hundred does not change the kind—what changes the kind is writing down one correspondence.

6. Roll Call and Closing

1. Four counts agree for $n = 0$ –8 (Computation 2.1): 1, 1, 2, 5, 14, 42, 132, 429, 1430.
2. $C_{10} = 16796$ (Observation 2.3): an independently known value.
3. $\binom{2n}{n}$ divides by $n + 1$ (Observation 2.4): checked for $n = 0$ –19.
4. The recurrence agrees too (Observation 2.5): an independent route.
5. $\binom{2n}{n}$ also agrees at $n = 0$ (Computation 3.1): 2 against 1 at $n = 1$.

6. A factor of 11 at $n = 10$ (Observation 3.3): identical if only the first is seen.
7. Asymptotic: $11.38 \rightarrow 1.13 \rightarrow 0.11$ per cent (Observation 3.4): approaching, never zero.
8. A bijection is stronger than agreement (Observation 4.2): it says which is which.
9. One agreement and all agreements differ in kind (Observation 5.3): counting gives no ordering.
10. What changes the kind is one correspondence (Observation 5.6): not the quantity.

Closing: Paper 300 wrote that numerical agreement alone is not taken as a common root——so what does establish one. With the Catalan numbers, four entirely different kinds of object give the same count——implementing parenthesis strings, binary trees, triangulations and lattice paths separately, all four agree in all nine rows from $n = 0$ to 8: 1, 1, 2, 5, 14, 42, 132, 429, 1430, matching $\binom{2n}{n}/(n+1)$ as well. $C_{10} = 16796$ matches the independently known value, $\binom{2n}{n}$ is divisible by $n+1$ for $n = 0$ to 19, and the recurrence $C_{n+1} = \sum_i C_i C_{n-i}$ holds for $n = 0$ to 12. Nine agreeing terms and two independent formulae——it looks sufficient. And it is still not evidence——the central binomial coefficient $\binom{2n}{n}$ also agrees at $n = 0$, both being 1, and parts at $n = 1$ with 2 against 1. At $n = 10$ the gap is exactly a factor of 11 (184756 against 16796), yet from the first term alone they look like the same sequence.

Whether one term agrees or nine, that is not evidence of a common root. The asymptotic form is more misleading still: it agrees nowhere and yet closes to 11.38 per cent at $n = 10$, 1.13 at $n = 100$ and 0.11 at $n = 1000$ ——as in Paper 388, approaching is not evidence. The four share a root because a bijection can be constructed——between strings and trees, strings and paths, and so on, there is a rule mapping each object to one object. A bijection is stronger than agreement of numbers: it contains the equality of counts and adds which corresponds to which, so the numerical agreement is a corollary. It therefore gives agreement at every n , and the number of checks stops mattering. The central binomial coefficient admits no such map: the agreement at $n = 0$ is only that both happen to be 1, and any attempt fails at $n = 1$. One thing does the separating——is the agreement over finitely many values, or is there a correspondence between the objects. The asymptotic agrees nowhere and resembles most closely; the central binomial agrees once and is wholly unrelated——resembling and sharing a root are entirely different, and counting agreements gives no ordering. The question “is there a bijection” decides it alone, with no numbers looked at. This is precisely what the criterion of Paper 300 says, and this paper sets an example that meets it and one that does not on the same page. To be honest——this paper constructs no bijection; it uses the existence of known ones, whose constructions are in the textbooks. What does not move is that agreement at $n = 0$ is not evidence and that they part at $n = 1$. One last thing——“nine terms agreed” is a quantity of evidence, not a kind. Ten or a hundred does not change the kind. What changes the kind is writing down one correspondence.

References

-
1. Stanley, R. P. (2015). *Catalan Numbers*. Cambridge: Cambridge University Press.
 2. Segner, J. A. (1758). Enumeratio modorum quibus figurae planae rectilineae per diagonales dividuntur in triangula. *Novi Commentarii Academiae Scientiarum Petropolitanae*.

3. Catalan, E. (1838). Note sur une équation aux différences finies. *Journal de Mathématiques Pures et Appliquées*.
4. Graham, R. L., Knuth, D. E., Patashnik, O. (1994). *Concrete Mathematics* (2nd ed.). Reading: Addison-Wesley.
5. Yamagishi, Y. (2026). *The Three Hundredth Census: Whether Two Things Share a Root Is Decidable* [Paper 300]. Zenodo. 10.5281/zenodo.22171914
6. Yamagishi, Y. (2026). *With a Million Candidates the Chance of Picking the Best Stays at **0.3679*** [Paper 388]. Zenodo. 10.5281/zenodo.22699147
7. Yamagishi, Y. (2026). *The Same One-Letter Typo Costs **15.75** Times More at One Position Than Another, and at the Second It Is Exactly **0*** [Paper 398]. Zenodo. 10.5281/zenodo.22699326
8. Yamagishi, Y. (2026). *The Four Hundredth Census: The Instrument Broke Before the Thing It Was Counting* [Paper 400]. Zenodo. 10.5281/zenodo.22699343

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)