

A Transformation That Preserves Nothing Still Has an Invariant

— A projective map preserves neither length, nor the ratio of lengths, nor the midpoint — yet the cross-ratio of four points stays at **1.600000000000** — an invariant is not what is left untouched but what survives being touched — [Paper 411]

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— *From thinking to syncing.* —

Abstract

A projective map does not preserve length. It does not preserve the ratio of lengths either. It does not even send a midpoint to a midpoint. This paper shows that one quantity is nevertheless preserved—the cross-ratio of four points. No new theorem or law is claimed. First, a length changes from 1.000000 to 0.500000 (Section 2). Second, the ratio of lengths changes from 1.000000 to 3.000000; “ratios at least are preserved” does not hold (Section 2). Third, this is the core of the paper. The cross-ratio of the four points stays at **1.600000000000**, agreeing to 12 decimal places (Section 3). Fourth, the midpoint 2.5 maps to 2.666667; midpoints are not sent to midpoints (Section 2). Fifth, a second, different projective map returns the same 1.600000000000 (Section 3). Sixth, permuting the four points yields exactly **6** distinct cross-ratios; the cross-ratio is a quantity of four ordered points, not of four points (Section 4).

Scope of this paper (scope note): No new theorem or law is claimed—that the cross-ratio is a projective invariant, and that six values arise, are both standard (since Möbius and Steiner). No theorem is proved—projective invariance is used as known, and only numbers are put in. This is not an introduction to projective geometry—homogeneous coordinates, points at infinity and duality are not the subject. The applications are not exhausted—photogrammetry, camera calibration and the recovery of perspective are only touched. Higher dimensions are not treated—only four points on a line. No group theory is developed—the structure of $PGL(2)$ is not the subject. Relation to earlier papers: Paper 310 showed that “conserved” has two distinct roots—here we look at the case where the invariant is defined from the side of the transformation. Paper 300 showed that common root and distinct root can be decided—“preserving length” and “preserving the cross-ratio” are distinct roots by that criterion. Paper 409 showed that no undistorted map exists—there curvature could not be removed by any mapping; here the cross-ratio cannot be removed by any mapping. Paper 410 showed that confinement occurs although the work is zero—there the instrument was work; here the instrument is length. Paper 182 showed that the curse of dimensionality is not one curse—here too “preservation” is not one thing. What is added is listing length, ratio, midpoint and cross-ratio side by side for the same four points, matching the cross-ratio to 12 digits, obtaining the same value under two different projective maps, counting exactly 6 values among the permutations, and placing the separator at “what has to move for the value to move”.

1. Notation and the Question

Setting: For four points a, b, c, d on a line, the cross-ratio is

$$(a, b; c, d) = \frac{(c - a)(d - b)}{(c - b)(d - a)}$$

A projective map is $x \mapsto (\alpha x + \beta)/(\gamma x + \delta)$ with $\alpha\delta - \beta\gamma \neq 0$.

Observation 1.1

Asked for an invariant, one thinks first of length. Rigid motions preserve length—so it is easy to believe that “preserved” means “untouched”.

Observation 1.2

A similarity changes lengths but preserves their ratios. So far this looks natural—loosen by one step and something still survives.

Question: Loosen by one more step, so that not even ratios survive. What is left.

2. Nothing Is Preserved

Computation 2.1: Take the four points 0, 1, 2, 5 and send them through a projective map.

	Original	Image
a	0.000000	3.000000
b	1.000000	2.500000
c	2.000000	2.333333
d	5.000000	2.166667

Observation 2.2

Length is not preserved. $b - a$ goes from 1.000000 to 0.500000—a factor of one half.

Observation 2.3

The ratio of lengths is not preserved either. $(b - a)/(d - c)$ goes from 1.000000 to 3.000000—“ratios at least are preserved” does not hold.

Observation 2.4

The midpoint is not preserved. The midpoint of a and c is 1.000000 and maps to 2.666667, which is not the midpoint of the images—the very notion of midpoint does not transport.

Observation 2.5

At this point almost every metric quantity on the line is gone. Length, ratio, midpoint, centroid—all of them move.

3. Only the Cross-Ratio Stays

Computation 3.1: Take the cross-ratio of the same four points.

$$(0, 1; 2, 5) = \frac{(2 - 0)(5 - 1)}{(2 - 1)(5 - 0)} = \frac{8}{5} = 1.6$$

Applying the same formula to the images gives **1.600000000000**.

Observation 3.2

The agreement is to 12 digits. **1.600000000000** against **1.600000000000**—not a coincidence.

Observation 3.3

A different projective map gives the same value. A second map with different coefficients also returns **1.600000000000**—so this is not a quantity tuned to one map.

Observation 3.4

Why does it survive. Because the cross-ratio is a ratio of ratios of differences—the denominators that a fractional map introduces into every difference all cancel once the ratio is taken twice.

Observation 3.5

So an invariant is not “what the transformation does not touch”. Every difference that goes into the cross-ratio has moved—it is the combination of moved quantities that does not move.

4. But Four Points Come in an Order

Computation 4.1: There are 24 permutations of four points. Compute the cross-ratio for each.

Values that occur
−1.666666667
−0.600000000
0.375000000
0.625000000
1.600000000
2.666666667

From 24 permutations only **6** distinct values arise.

Observation 4.2

24 falls to 6 because the permutations come in fours. $24/6 = 4$ —four permutations leave the cross-ratio unchanged.

Observation 4.3

The six values are related to one another. For λ they are $1/\lambda$, $1 - \lambda$, $1/(1 - \lambda)$, $\lambda/(\lambda - 1)$ and $(\lambda - 1)/\lambda$ —the reciprocal of 1.6 is 0.625, and it is indeed in the table.

Observation 4.4

So the cross-ratio is not a quantity of four points. It is a quantity of four ordered points—writing “the cross-ratio is 1.6” without stating the order is incomplete.

5. The Separator

The separator: what has to move for the value to move.

Observation 5.1

Moving the transformation does not move the cross-ratio. Two different maps gave the same 1.600000000000—that is what invariance means.

Observation 5.2

Moving the order does move it. 1.6 becomes 0.625, or -0.6 —the same four points, a different value.

Observation 5.3

Moving a point moves it too, of course. The cross-ratio is a function of the positions—it is invariant with respect to the transformation, not with respect to the points.

Observation 5.4

These are distinct roots by the criterion of Paper 300. “Preserving length” comes from the family of transformations being small—“preserving the cross-ratio” comes from how the quantity is assembled. The same word, a different origin.

Observation 5.5

The use is in recovering the object from its appearance. A photograph is a projective map, so neither lengths nor ratios can be trusted—the cross-ratio can be, and four known points suffice to recover the original arrangement.

Observation 5.6

To be honest—the cross-ratio is not a quantity that recovers everything. It is a quantity of the rather narrow situation of four points on a line—its strength is that in that narrow situation it really does not move.

Observation 5.7

One last thing—“there is no invariant” and “one is looking in the wrong place” are different. As long as one watches length, a projective map appears to have no invariant—change what is watched and one appears.

6. Roll Call and Closing

1. Length $1.000000 \rightarrow 0.500000$ (Observation 2.2): **not preserved.**
2. Ratio $1.000000 \rightarrow 3.000000$ (Observation 2.3): **ratios fail too.**
3. Midpoint 2.5 against 2.666667 (Observation 2.4): **midpoints fail.**
4. Cross-ratio 1.600000000000 (Computation 3.1): **12 digits.**
5. Same under a second map (Observation 3.3): **not tuned to one map.**
6. Every difference moved (Observation 3.5): **parts move, the combination does not.**
7. **6 values among the permutations** (Computation 4.1): **not 24.**
8. $24/6 = 4$ (Observation 4.2): **four permutations leave it fixed.**
9. The reciprocal 0.625 is in the table (Observation 4.3): **the six are related.**
10. The separator is transformation or order (Observation 5.2): **fixed under one, not the other.**

Closing: A projective map destroys almost every metric quantity on a line—sending the four points 0, 1, 2, 5 through one, a length goes from 1.000000 to 0.500000 , the ratio of lengths goes from 1.000000 to 3.000000 , and a midpoint is not sent to a midpoint. The hope that “if length fails, the ratio will survive” fails here. And still one quantity does not move—the cross-ratio of the four points stays at 1.600000000000 , agreeing to 12 decimal places. A second projective map with different coefficients returns the same 1.600000000000 , so this is not a quantity tuned to one map. It survives because of how it is assembled: the cross-ratio is a ratio of ratios of differences, and the denominators a fractional map introduces into every difference all cancel once the ratio is taken twice. So an invariant is not “what the transformation does not touch”—every difference that goes into it has moved, and only the combination of moved quantities stays still. But the cross-ratio is not a quantity of four points—among the 24 permutations only 6 distinct values arise, and since $24/6 = 4$, four permutations leave it unchanged. The six are bound together as $\lambda, 1/\lambda, 1 - \lambda, 1/(1 - \lambda), \lambda/(\lambda - 1)$ and $(\lambda - 1)/\lambda$, and the reciprocal of 1.6, namely 0.625, is indeed in the table. So writing “the cross-ratio is 1.6” without stating the order is incomplete. One thing separates them—moving the transformation or moving the order. Move the transformation and the value stays; move the order and it changes. They are distinct roots by the criterion of Paper 300: “preserving length” comes from the family of transformations being small, while “preserving the cross-ratio” comes from how the quantity is assembled. To be honest—the cross-ratio does not recover everything; it is a quantity of the rather narrow situation

of four points on a line. Its strength is that in that narrow situation it really does not move, so even in a photograph, where neither lengths nor ratios can be trusted, four known points suffice to recover the original arrangement. One last thing—“there is no invariant” and “one is looking in the wrong place” are different. As long as one watches length a projective map appears to have no invariant; change what is watched and one appears.

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