

An Area Is Not a Solid Angle

— The Hopf Factorisation Holds at Every Radius, and $R = 2$ Is a Choice of Unit — [Paper 45] 3rd ed.

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— *From thinking to syncing.* —

ABSTRACT

This paper treats the normalisation factor $1/(16\pi^2)$ of the topological charge and made three claims: (A) that it descends from the component form of the second Chern class, (B) that $16\pi^2$ is the volume of the three-sphere of radius $R = 2$, with base and fibre both equal to 4π in the Hopf factorisation, and (C) that the instanton action can be written $S = 1/(2\alpha_{\text{eff}})$. (A) is correct and is not altered here. What the third edition corrects is (B) and (C). First, the Hopf factorisation itself is genuine, and it does not require $R = 2$. With $\text{Vol}(S_R^3) = 2\pi^2 R^3$, the base area $= 4\pi(R/2)^2 = \pi R^2$ and the fibre circumference $= 2\pi R$, their product equals the total volume for every R (verified here symbolically). This is a genuine geometric fact and needs no specification of the radius. Second, “ $R = 2$ is the unique radius” is not geometry but a choice of unit. The second edition solved $\pi R^2 = 2\pi R$ to get $R = 2$. But the left side is an area (dimension L^2) and the right side a circumference (dimension L^1), and equating them requires fixing a unit for R first. Rescaling by $R = \lambda R'$ gives $R' = 2/\lambda$ (verified here): $\lambda = 2$ gives $R' = 1$, and $\lambda = 10$ gives $R' = 1/5$. What “ $R = 2$ ” states is which unit was chosen (Paper 95). Third, and the largest correction here: the 4π of the base is not a solid angle. The solid angle of any sphere is 4π steradian — independent of radius, and dimensionless. The area of $S_{R/2}^2$, by contrast, is πR^2 with dimension L^2 : it is π at $R = 1$, 4π at $R = 2$ and 9π at $R = 3$ (verified here). The agreement is a numerical coincidence at $R = 2$, and calling the area a “solid angle” hides that difference (Paper 16 on the distinct roots of the equals sign; Paper 116 on judging by dimension). Corollary 2.3, “the double appearance of the Yamagishi patch is a geometric necessity”, is therefore retracted. Fourth, $S_{\text{inst}} = 1/(2\alpha_{\text{eff}})$ is not a theorem but a restatement of a definition. Writing $\alpha_{\text{eff}} \equiv g^2/X$ in general gives $1/(2\alpha_{\text{eff}}) = X/(2g^2)$, and this equals $8\pi^2/g^2$ precisely when $X = 16\pi^2$ — that is, it holds because $16\pi^2$ was placed inside the definition of α_{eff} (verified here symbolically). The content of the identity is the single line $8\pi^2 = 16\pi^2/2$ (the same shape as $l_p/t_p = c$ in Paper 40). Fifth, whether the two $16\pi^2$ share a root is undecided. The loop-measure side gives $2\pi^2/(2\pi)^4 = 1/(8\pi^2)$ from the angular part $\text{Vol}(S_1^3) = 2\pi^2$ of $\int d^4k/(2\pi)^4$; the topological side gives $8\pi^2|k|/g^2$ from the BPS bound. Both descend from four-dimensional integration, but the appearance of $16\pi^2$ in each is not a proof of a common root (Papers 86, 100). Sixth, the Hopf picture does no work in the instanton count. The second edition rightly corrected the classification to $\pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$ rather than the Hopf map's $\pi_3(S^2)$. Both are $\mathbb{Z}$, so the value is unchanged while the reason changes — and after that correction the Hopf factorisation is not doing anything for this paper's subject. The section on the strong CP problem (second edition) is correct and is kept unchanged: it delimits the initialisation-condition approach as partial, since the second term of $\theta_{\text{eff}} = \theta_{\text{QCD}} + \arg \det M$ remains unsolved. Scope: structural interpretation and self-audit of established standard facts (Layer 2). The strong CP problem is not solved; no new theorem and no prediction. The cosmic-OS lens is quarantined under a non-scientific label.

Standing of this paper (discipline): the mathematics and physics used here are **entirely standard** — Chern–Weil theory, the BPST instanton (1975), the Hopf bundle, $\pi_3(S^3) = \mathbb{Z}$, the nEDM measurement, and the lattice-QCD result $m_u \neq 0$. **Not one new theorem is proved.** The third edition's contribution is **a sorting by dimension:** (i) showing the Hopf factorisation holds **at every R** , (ii) showing that “ $R = 2$ is unique” is **a choice of unit**, by rescaling, (iii) judging by dimension that the base's 4π is **an area and not a solid angle**, (iv) showing by generalisation that $S = 1/(2\alpha_{\text{eff}})$ is **a restatement of a definition**, and (v) placing the two $16\pi^2$ in the category “**common root undecided**”. **The most important restraints: an area and a solid angle are not put in the same column** (different dimensions; Papers 16, 116). $R = 2$ is **not called a geometric necessity** (a choice of unit; Paper 95). **A restatement of a definition is not called a theorem** (Papers 40, 21, 16). **The two $16\pi^2$ are not declared to share a root** (Papers 86, 100). **No claim is made to have solved the strong CP problem** ($\arg \det M$ is unsolved; the second edition's delimitation is kept). **No alternative to the axion or to Nelson–Barr models is proposed.** The cosmic-OS lens is **Layer 3** and is quarantined under a non-scientific label.

1. Introduction: The Paper Made Three Claims

(A) Theorem 2.1	$1/(16\pi^2)$ descends from the second Chern class
(B) Theorem 2.2, Cor. 2.3	$16\pi^2 = \text{Vol}(S^3_{R=2})$; $R = 2$ a geometric necessity
(C) Theorem 3.1	$S_{\text{inst}} = 8\pi^2/g^2 = 1/(2\alpha_{\text{eff}})$

(A) is correct. Writing the second Chern class $c_2 = \frac{1}{8\pi^2} \text{Tr}(F \wedge F)$ in components gives $k = \frac{1}{16\pi^2} \int d^4x \text{Tr}(F_{\mu\nu} \tilde{F}^{\mu\nu})$. It is standard and is **not altered here**.

What the third edition corrects is (B) and (C).

Both are corrections of a kind visible from the dimensions.

2. The Hopf Factorisation Is Genuine — and Needs No $R = 2$

Theorem 2.1 (The Hopf factorisation holds at every radius)

For the Hopf bundle $S^1 \hookrightarrow S_R^3 \rightarrow S_{R/2}^2$,

$$4\pi\left(\frac{R}{2}\right)^2 \times 2\pi R = \pi R^2 \times 2\pi R = 2\pi^2 R^3 = \text{Vol}(S_R^3).$$

Here $4\pi(R/2)^2$ is the **base area** and $2\pi R$ the **fibre circumference**. This holds for every R (verified here symbolically).

Observation 2.1 (This is where the real content lies)

That the volume of the total space factorises into the area of the base times the length of the fibre — this is the genuine geometric fact about the Hopf bundle, and it requires no specification of a radius. The second edition drew attention to the number $16\pi^2$ that appears on substituting $R = 2$. But the factorisation itself does not depend on R . $R = 2$ is only how that fact looks in one particular scale.

3. $R = 2$ Is a Choice of Unit

The second edition wrote: “ $R = 2$ is the unique radius at which the solid angle of the base and the circumference of the fibre coincide, so the double appearance of the Yamagishi patch is a geometric necessity”.

Theorem 3.1 (The equation is meaningful only once a unit is fixed)

Solving $\pi R^2 = 2\pi R$ gives $R = 2$. But the left side has dimension L^2 and the right side dimension L^1 . Rescaling by $R = \lambda R'$,

$$\pi(\lambda R')^2 = 2\pi(\lambda R') \implies R' = \frac{2}{\lambda}$$

(verified here).

Scale λ	“the unique radius”
1	$R' = 2$
2	$R' = 1$
1/2	$R' = 4$
10	$R' = 1/5$

Observation 3.1 (“ $R = 2$ ” states which unit was chosen)

Joining two quantities of different dimension with an equals sign has no meaning until a unit has been fixed. And once the unit changes, so does “the unique radius”. What changes its answer when the scale changes is not a geometric necessity (Paper 95). Corollary 2.3, “the double appearance of the Yamagishi patch is a geometric necessity”, is therefore retracted. What is retracted is the necessity, not the factorisation — Theorem 2.1 of this edition stands.

4. An Area Must Not Be Called a Solid Angle

This is the largest correction in this edition.

The second edition called the base's πR^2 “the solid angle of the base”. It is not a solid angle.

Established Fact 4.1 (A solid angle and an area are different kinds of quantity)

The **solid angle** of any sphere is 4π steradian — **independent of radius and dimensionless**. The **area** is $4\pi r^2$, of dimension L^2 .

R	Area of $S^2_{R/2} = \pi R^2$	Solid angle of $S^2_{R/2}$
1	π	4π
2	4π	4π
3	9π	4π

Observation 4.1 (The agreement is a numerical coincidence at $R = 2$)

The **solid-angle column does not move**. It is 4π at every radius. The **area column moves as R^2** . They take the same value at the single point $R = 2$, and **that is a coincidence of numbers, not an identity of quantities**. Calling the area a “solid angle” erases that difference. And the moment it is erased, a **quantity independent of R (the solid angle) stands in the same column as a quantity that depends on R (the area)** — precisely the error this series named in Paper 16 (“the equals sign has distinct roots”) and in Paper 116 (“quantities of different dimension cannot be the same quantity”). **What the second edition read as a “geometric necessity” was this renaming.**

5. $S_{\text{inst}} = 1/(2\alpha_{\text{eff}})$ Is a Restatement of a Definition

The second edition called Theorem 3.1 “**the core result of this paper**” and named it the **perturbative--non-perturbative duality theorem**.

Theorem 5.1 (Generalising makes the content visible)

Writing $\alpha_{\text{eff}} \equiv g^2/X$ in general gives $\frac{1}{2\alpha_{\text{eff}}} = \frac{X}{2g^2}$. This equals the BPST value $S_{\text{inst}} = \frac{8\pi^2}{g^2}$ precisely when

$$\frac{X}{2} = 8\pi^2 \iff X = 16\pi^2$$

(verified here symbolically).

Observation 5.1 (The content of the identity is one line)

$S = 1/(2\alpha_{\text{eff}})$ holds because $16\pi^2$ was placed inside the definition of α_{eff} . Put otherwise, everything this “theorem” states is $8\pi^2 = \frac{16\pi^2}{2}$ and no more. A definition was made, and then the definition was used to rewrite the expression. This is the same shape Paper 40 found inside itself — $l_p/t_p = c$ holds because $\hbar G$ cancels, and verifying it numerically confirms nothing. An equation with no violable null hypothesis is not called a theorem (Papers 21, 16). The name “perturbative–non-perturbative duality theorem” is therefore retracted and rewritten as “a restatement of a definition”. The algebra is right. What changed is the classification of what it is.

6. The Two $16\pi^2$: Common Root Undecided

The second edition stated that “perturbation theory and non-perturbative effects are governed by the same geometric quantity $16\pi^2$ ”. This is not retracted but marked undecided.

Loop-measure side	angular part of $\int \frac{d^4 k}{(2\pi)^4}$ is $\text{Vol}(S_1^3) = 2\pi^2$; $\frac{2\pi^2}{(2\pi)^4} = \frac{1}{8\pi^2}$
Topological side	$\int \text{Tr}(F\tilde{F}) = 16\pi^2 k$; the BPS bound gives $S = \frac{8\pi^2 k }{g^2}$

Observation 6.1 (Appearing in both is not a proof of a common root)

Both descend from four-dimensional integration — one from the measure on momentum space, the other from the second Chern class. In that sense they are not unrelated. But to say they are “governed by the same geometric quantity” requires grounds beyond the appearance of $16\pi^2$ in each. This series wrote in Paper 86 that a common ancestor is not a common root, and in Paper 100 that a resonance is a signature, not a proof. This edition therefore records the matter as undecided, calling it neither a common root nor a rhyme.

7. The Hopf Picture Does No Work in the Instanton Count

The second edition made one correct correction: instantons are classified by $\pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$, not by the Hopf map's $\pi_3(S^2)$.

Observation 7.1 (Same value, different reason — and the picture does not survive)

Both $\pi_3(S^2) = \mathbb{Z}$ and $\pi_3(S^3) = \mathbb{Z}$. The conclusion that the instanton number is an integer is therefore unchanged; what changed is the reason. And once that correction is accepted, the Hopf factorisation is doing nothing for this paper's subject. The origin of $16\pi^2$ is the second Chern class (claim (A)), and it does not pass through the Hopf bundle. The factorisation of Section 2 is correct as geometry. But this paper does not need it. It is the same shape Paper 41 found inside itself — two correct results welded together by a title.

8. The Strong CP Problem: The Second Edition's Delimitation Is Kept

Established Fact 8.1 (What the second edition stated correctly)

What observation constrains is $\theta_{\text{eff}} = \theta_{\text{QCD}} + \arg \det M$, not θ_{QCD} . In the Standard Model CP is in fact violated, and the CKM matrix carries an $O(1)$ complex phase, so $\arg \det M$ is generically $O(1)$. Hence imposing $\theta_{\text{QCD}} = 0$ still leaves $\theta_{\text{eff}} \sim O(1) \gg 10^{-10}$. The real difficulty of the strong CP problem is not “setting θ to zero” but explaining why two quantities of independent origin cancel to one part in 10^{10} . The initialisation-condition approach addresses the former and does not answer the latter.

This delimitation is accurate and is kept unchanged in the third edition. The comparison with existing approaches (Peccei--Quinn; $m_u = 0$, excluded by lattice QCD; Nelson--Barr) is also unchanged.

Only one thing changes: the cosmic-OS lens that the second edition placed in the body (a phase offset in the base memory cells, a clean format) is removed from the body as Layer 3. The content of the delimitation never depended on that lens.

9. The Scope, Restated

Item	What it is
$1/(16\pi^2)$ from the second Chern class	Established fact (unchanged)
$\text{Vol}(S_R^3) = \text{base area} \times \text{fibre length}$	Geometric fact (every R)
$R = 2$ the unique radius	Choice of unit ($R' = 2/\lambda$)
The base's 4π as a “solid angle”	Retracted (solid angle dimensionless; area L^2)
The patch's double appearance as necessity	Retracted
$S_{\text{inst}} = 1/(2\alpha_{\text{eff}})$	Restatement of a definition ($8\pi^2 = 16\pi^2/2$)
The two $16\pi^2$ as one geometric quantity	Undecided (neither root nor rhyme declared)
The Hopf bundle giving the instanton number	Not shown (it is $\pi_3(SU(2))$; corrected already)
A solution of the strong CP problem	Not shown ($\arg \det M$ unsolved)

Of the nine rows, two are settled results — the origin of $1/(16\pi^2)$, and the Hopf factorisation. And the latter does no work for this paper's subject.

10. Conclusion

This paper was looking for the geometric substance of $16\pi^2$. What it found is genuine: the total space of the Hopf bundle factorises into the area of the base times the length of the fibre. At every radius.

What looked like necessity was a scale. $R = 2$ is unique because an area was joined to a length by an equals sign, and that sign cannot be written until a unit has been fixed. Double the scale and the unique radius becomes 1.

And what was called “the solid angle of the base” was an area. The solid angle of any sphere is 4π — and when a quantity that does not move is placed beside one that does, an agreement looks like a necessity.

A solid angle does not know the radius. An area does.
There simply happened to be a radius at which the two took the same number.

What is not claimed (restraint): an area and a solid angle are not treated as the same quantity (different dimensions; Papers 16, 116) / $R = 2$ is not called a geometric necessity (a choice of unit; Paper 95) / a restatement of a definition is not called a theorem (Papers 40, 21, 16) / the two $16\pi^2$ are not declared to share a root (undecided; Papers 86, 100) / the Hopf bundle is not said to classify the instanton number (it is $\pi_3(SU(2))$) / no claim is made to have solved the strong CP problem ($\arg \det M$ is unsolved; the second edition's delimitation is kept) / no alternative to the axion or to Nelson--Barr models is proposed / the cosmic-OS lens is not placed in the body (Layer 3, quarantined) / no new mathematics and no prediction (Layer 2).

Revision Record

Third edition (August 2026): Theorem 2.1 (that $1/(16\pi^2)$ descends from the component form of the second Chern class) and the second edition's delimitation of the strong CP problem are unchanged. The corrections are three, all by sorting on dimension. (i) It was made explicit that the Hopf factorisation holds at every R : $4\pi(R/2)^2 \times 2\pi R = 2\pi^2 R^3 = \text{Vol}(S_R^3)$ (verified symbolically). The real content is that factorisation, and it needs no specification of the radius. (ii) “ $R = 2$ is the unique radius” was rewritten as a choice of unit: $\pi R^2 = 2\pi R$ equates an area (L^2) with a circumference (L^1) and has no meaning until a unit for R is fixed. Rescaling by $R = \lambda R'$ gives $R' = 2/\lambda$ ($R' = 1$ at $\lambda = 2$; $R' = 1/5$ at $\lambda = 10$). Corollary 2.3, “the double appearance of the Yamagishi patch is a geometric necessity”, is therefore retracted. (iii) The description of the base's 4π as a “solid angle” was retracted: the solid angle of any sphere is 4π steradian, radius-independent and dimensionless, while the area is πR^2 of dimension L^2 — π at $R = 1$, 4π at $R = 2$, 9π at $R = 3$, so the agreement is a coincidence of numbers at $R = 2$. (iv) Theorem 3.1 was renamed from “the perturbative--non-perturbative duality theorem” to “a restatement of a definition”: generalising to $\alpha_{\text{eff}} \equiv g^2/X$ shows that $1/(2\alpha_{\text{eff}}) = 8\pi^2/g^2$ requires $X = 16\pi^2$, so it holds because $16\pi^2$ was placed inside the definition (verified symbolically). The content is the single line $8\pi^2 = 16\pi^2/2$. The algebra is right; the classification changed. (v) The assertion that the two $16\pi^2$ are “the same geometric quantity” was changed to “undecided”: the loop side comes from $\text{Vol}(S_1^3) = 2\pi^2$ and $(2\pi)^4$, the topological side from the BPS bound. Both descend from four-dimensional integration, but appearing in each is not a proof of a common root (Papers 86, 100). (vi) Following the second edition's correct correction (the classification is $\pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$), it was made explicit that the Hopf factorisation does no work for this paper's subject. (vii) The cosmic-OS lens was removed from the body (Layer 3). **Terminological correction (August 2026):** the phrase “Hopf decomposition” has been replaced throughout by “Hopf factorisation”. The object referred to is unchanged — the factorisation in which the base area times the fibre circumference equals $\text{Vol}(S_R^3)$. The reason for the change is that “Hopf decomposition” collides with Eberhard Hopf's term in ergodic theory (the splitting into conservative and dissipative parts), as noted in Paper 125.

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