

That Everything Cannot Be Tuned at Once Is a Proved Fact

— Twelve perfect fifths overshoot seven octaves by **23.4600** cents — equal temperament divides that by twelve and holds the fifth to **1.9550** — but the major third is off by **−13.6863** cents, **7.0007** times as far —

[Paper 466]

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— *From thinking to syncing.* —

Abstract

Equal temperament is called a compromise. This paper shows that the necessity of compromising is settled by arithmetic— $(3/2)^{12}$ and 2^7 disagree because 2 and 3 are coprime. No new theorem or law is claimed. First, twelve fifths overshoot seven octaves by **23.4600** cents (Section 2). Second, the **1.9550** cent error of the tempered fifth is exactly one twelfth of that comma (Section 2). Third, this is the core of the paper. The major third is off by **−13.6863** cents, **7.0007** times the fifth (Section 3). Fourth, the comma on the side of the third is **21.5063** cents, a different quantity from **23.4600** (Section 3). Fifth, equal temperament fixes the fifth and sacrifices the third (Section 3). Sixth, the separator is whether this is approximation error or evidence that the circle does not close (Section 4).

Keywords: musical tuning, equal temperament, Pythagorean comma, syntonic comma, just intonation.

Scope of this paper (scope note): No new theorem or law is claimed—the Pythagorean comma, the syntonic comma and twelve-tone equal temperament are all standard. No music is judged—which temperament is beautiful and how to perform are not the subject. Hearing is not discussed—beats, thresholds and consonance are not the subject; only the arithmetic of ratios. History is excluded—meantone, Werckmeister, Kirnberger and their circumstances are not the subject. Instruments are not discussed—tuning practice, inharmonicity and adaptive tuning are not the subject. Relation to earlier papers: Paper 465 showed that size acts forty times harder than material—there too, the relative size of two factors was the question. Paper 300 showed that common root and distinct root can be decided —“the comma of the fifth” and “of the third” are distinct roots by that criterion. Paper 459 showed that a round trip multiplies the loss by 1.9—there too, repetition became a power. What is added is giving the Pythagorean comma as **23.4600** cents and matching the known value, showing the tempered fifth’s error to be exactly one twelfth of it, giving the third’s error as **−13.6863** cents, giving the ratio as **7.0007**, showing that the two commas are different, and placing the separator at approximation versus non-closure.

1. Notation and the Question

Setting: Measure an interval by the logarithm of the frequency ratio, in cents,

$$c = 1200 \log_2 \frac{f_2}{f_1}$$

so that an octave is 1200 cents. A just perfect fifth is the ratio $3/2$ and a just major third is $5/4$. In twelve-tone equal temperament the fifth is $2^{7/12}$ and the major third $2^{4/12}$.

Observation 1.1

Equal temperament is called a compromise. Every key playable alike, and every chord a little off—that is the usual account.

Observation 1.2

But “compromise” sounds like something that better work could avoid. It cannot be avoided—and the reason is arithmetic.

Question: By how much, and where is the error put.

2. Twelve Fifths Overshoot by 23.4600 Cents

Computation 2.1: Compare twelve just fifths with seven octaves.

$(3/2)^{12}/2^7 = 1.013643265$, that is **23.4600** cents.

Observation 2.2

It matches the known Pythagorean comma of 23.46 cents. An independent check—and exceeding 1, the circle does not close.

Observation 2.3

It does not close because 2 and 3 are coprime. $3^{12} = 531441$ and $2^{19} = 524288$ never agree—no positive integers give $2^a = 3^b$.

Computation 2.4: Take the error of the tempered fifth against the just fifth.

1.9550 cents (just 1.500000, tempered 1.498307).

Observation 2.5

That is exactly one twelfth of the comma. $23.4600/12 = 1.9550$ exactly—equal temperament is the act of dividing the non-closure by twelve and distributing it.

Observation 2.6

1.9550 cents is barely audible. As far as fifths go, equal temperament is very good—the trouble lies elsewhere.

3. The Third Is 7.0007 Times Further Off

Computation 3.1: Take the error of the tempered major third.

Interval	Just	Error against tempered [cents]
Perfect fifth	1.500000	1.9550
Major third	1.250000	−13.6863

Observation 3.2

The third is **7.0007** times further off than the fifth. 13.6863 against 1.9550—within one temperament, the accuracy differs entirely by interval.

Observation 3.3

The sign is reversed. The tempered fifth is flat and the tempered third is sharp—which is what the minus sign records.

Computation 3.4: Take the comma on the side of the third, the syntonic comma.

The ratio of $(3/2)^4/2^2$ to $5/4$ is **21.5063** cents.

Observation 3.5

It matches the known 21.51 cents. An independent check—and it differs from the Pythagorean comma of 23.4600 by 1.9537 cents.

Observation 3.6

So there are two places where the arithmetic fails to close. One between 2 and 3, one between 2, 3 and 5—separate discrepancies from separate primes.

Observation 3.7

Equal temperament divides the first by twelve and leaves the second alone. Hence 1.9550 for the fifth and 13.6863 for the third—the result of a choice about which to fix.

4. The Separator

The separator: approximation error, or evidence that the circle does not close.

Observation 4.1

Approximation error can be reduced by precision. Compute more finely—no wall in principle.

Observation 4.2

Non-closure survives any precision. Because no positive integers satisfy $2^a = 3^b$ —23.4600 cents is a consequence of arithmetic, not of measurement.

Observation 4.3

These are distinct roots by the criterion of Paper 300. Both are called “the error of a temperament”—one the incommensurability of 2 and 3, one of those with 5 added.

Observation 4.4

So searching for a better temperament cannot zero both. It only chooses which to fix—meantone fixed the third and sacrificed the fifth; equal temperament did the reverse.

Observation 4.5

This stands where Paper 459 stands. There, efficiency became a power when stages were stacked—here fifths stacked become a power that never meets a power of 2. In both, repetition builds an exponent.

Observation 4.6

It overlaps Paper 465 as well. There, the relative size of two factors settled the answer—here the relative size of two commas settles which is sacrificed.

Observation 4.7

To be honest—nothing is said about which errors are audible. How 13.6863 cents actually sounds depends on chord, register, timbre and beating—what can be said here is only that the error is arithmetically unavoidable.

5. Roll Call and Closing

1. The comma is **23.4600** cents (Computation 2.1): **matching the known value.**
2. Because **2** and **3** are coprime (Observation 2.3): **$3^{12} \neq 2^{19}$.**
3. The fifth is off by **1.9550** (Computation 2.4): **one twelfth of the comma.**
4. Exactly (Observation 2.5): **divided and distributed.**
5. The third is off by **−13.6863** (Computation 3.1): **the core.**
6. **7.0007** times the fifth (Observation 3.2): **accuracy differs by interval.**
7. The sign is reversed (Observation 3.3): **the tempered third is sharp.**
8. The syntonic comma is **21.5063** (Computation 3.4): **a different quantity.**
9. The separator is error or non-closure (Observation 4.3): **distinct roots by Paper 300.**
10. Both cannot be zeroed (Observation 4.4): **only a choice remains.**

Closing: Equal temperament is called a compromise—every key playable alike and every chord a little off. But “compromise” sounds like something better work could avoid. It cannot be avoided, and the reason is arithmetic. Non-closure is a consequence of arithmetic—twelve just fifths overshoot seven octaves by a factor of **1.013643265**, that is **23.4600** cents, matching the known Pythagorean comma. It does not close because **2** and **3** are coprime: $3^{12} = 531441$ and $2^{19} = 524288$ never agree, and no positive integers give $2^a = 3^b$. The **1.9550** cent error of the tempered fifth is exactly one twelfth of that comma—equal temperament is the act of dividing the non-closure by twelve and distributing it. And **1.9550** cents is barely audible, so as far as fifths go equal temperament is very good. The trouble is in the third—the major third is off by **−13.6863** cents, **7.0007** times the fifth. Within one temperament the accuracy differs entirely by interval. And the sign is reversed: the tempered fifth is flat while the tempered third is sharp. The comma on that side, the syntonic comma, is **21.5063** cents, matching the known value and differing from the Pythagorean comma by **1.9537** cents. So there are two places where the arithmetic fails to close, one between **2** and **3** and one between **2**, **3** and **5**: separate discrepancies from separate primes. Equal temperament divides the first by twelve and leaves the second alone, which is why the fifth is off by **1.9550** and the third by **13.6863**—the result of a choice about which to fix. One thing separates them—approximation error, or evidence that the circle does not close. Approximation error can be reduced by precision, with no wall in principle. Non-closure survives any precision, because no positive integers satisfy $2^a = 3^b$: **23.4600** cents is a consequence of arithmetic, not of measurement. They are distinct roots by the criterion of Paper 300, one the incommensurability of **2** and **3** and one of those with **5** added. So searching for a better temperament cannot zero both; it only chooses which to fix, as meantone fixed the third and equal temperament the fifth. This stands where Paper 459 stands, where stacking stages made efficiency a power: here stacking fifths makes a power that never meets a power of **2**. It overlaps Paper 465 too, where the relative size of two factors settled the answer: here two commas settle which is

sacrificed. To be honest—nothing is said about which errors are audible. How 13.6863 cents actually sounds depends on chord, register, timbre and beating; what can be said here is only that the error is arithmetically unavoidable.

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