

The Pfaffian's $2^n n!$ Is One Block

— 3rd ed.: Where the Yamagishi Euler-Characteristic Unpacking Cut $(4\pi)^n n!$ — [Paper 48]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026 (Third Edition)

— *From thinking to syncing.* —

ABSTRACT

This paper showed that the normalisation constant of the Chern--Gauss--Bonnet theorem in $2n$ dimensions is $(4\pi)^n n!$, and settled that **the factor of 2 in $32\pi^2 = 2 \times 16\pi^2$ is the $n! = 2!$ of the Pfaffian's antisymmetrisation. That settlement is correct and is not altered here** — neither the first edition's “double buffering” nor Paper 32's “dimension of the $SU(2)$ fundamental / double cover” is reinstated. **What the third edition corrects is the way $(4\pi)^n n!$ was cut. First, following the derivation factor by factor puts the seam elsewhere.** The Euler class is $e(TM) = \frac{1}{(2\pi)^n} \text{Pf}(\Omega)$ and the Pfaffian is $\text{Pf}(\Omega) = \frac{1}{2^n n!} \epsilon^{a_1 \dots a_{2n}} \Omega_{a_1 a_2} \wedge \dots$, so the normalisation constant is

$$(2\pi)^n \times (2^n n!)$$

(verified here for $n = 1, 2, 3, 4$), and $2^n n!$ is one block — the Pfaffian's antisymmetrisation factor. The second edition tore the 2^n out of that block and glued it to $(2\pi)^n$ to form $(4\pi)^n$. Second, this contradicts the second edition's own finding. It correctly identified “ $n!$ as the combinatorial factor of the Pfaffian”. But 2^n comes from the same place: $\frac{1}{2^n n!}$ is a single denominator in the definition of the Pfaffian, counting the ways of splitting $2n$ indices into n antisymmetric pairs — $n!$ for the ordering of the pairs, 2^n for the swap within each pair. **One cannot assign $n!$ to combinatorics while handing 2^n to geometry. Third, there are two standings. $(2\pi)^n$ is the normalisation that makes the periods of characteristic classes integral (the Chern--Weil convention); $2^n n!$ is combinatorics. The quantity $(4\pi)^n$ is regrouped across those two, and the 4π so formed is not a solid angle (the same shape as Papers 45 and 46). Fourth, one line of Section 4 is therefore corrected. The second edition concluded that “what the three share is not the factor of 2 but the substrate $(4\pi)^n$ ”. The judgement that the three factors of 2 have three different origins is correct and is kept — combinatorics for the Euler characteristic, component expansion for the Chern class, generator normalisation for the θ term. What is corrected is the name of the substrate: what they share is not $(4\pi)^n$ but the (2π) of the characteristic-class normalisation. All numbers are unchanged — 4π in two dimensions, $32\pi^2$ in four, $384\pi^3$ in six, and the two-dimensional consistency with classical Gauss--Bonnet via $R = 2K$ (re-verified here). **Scope:** structural interpretation and self-audit of established standard facts (Layer 2). No new theorem and no prediction. The cosmic-OS lens is quarantined under a non-scientific label.**

Standing of this paper (discipline): the mathematics used here is **entirely standard** — Chern's Gauss--Bonnet (1944, 1945), the Euler class and the Pfaffian, the normalisation of characteristic classes. **Not one new theorem is proved.** The third edition's contribution is **putting the seam in the right place:** (i) exhibiting the normalisation constant factor by factor as $(2\pi)^n \times (2^n n!)$, (ii) stating that $2^n n!$ is **one block of the Pfaffian**, (iii) making explicit that $(4\pi)^n$ is formed across that block, and (iv) correcting Section 4's “shared substrate” from $(4\pi)^n$ to $(2\pi)^n$. **The most important restraints: 2^n is not separated from $n!$** (they are the same Pfaffian denominator; the same shape as the prohibition on splitting modules in Papers 13, 14). **The 4π of $(4\pi)^n$ is not called a solid angle** (Papers 45, 46). **The three origins of the factor 2 are not merged** (the second edition's judgement is kept; Papers 86, 100). **The first edition's “double buffering” and Paper 32's “ $SU(2)$ dimension / double cover” are not reinstated** (retracted in the second edition). **No new theorem about the Euler characteristic is asserted.** Layer-3 lenses are quarantined under a non-scientific label.

1. Introduction: The Settlement Was Right; What Is Corrected Is the Cut

The second edition settled a standing question — **the factor of 2 in $32\pi^2 = 2 \times 16\pi^2$ is the combinatorial factor $n! = 2!$ of the Pfaffian's antisymmetrisation** — and retracted both the first edition's “front / back double buffering” and Paper 32's conjecture of “the dimension of the $SU(2)$ fundamental, or the double cover S^3 of S^2 ”.

That settlement is correct and is not reinstated against here.

But in writing the settlement down, the second edition shifted the seam by one factor.

2. Following the Derivation Factor by Factor

Theorem 2.1 (The natural decomposition of the normalisation constant)

The Euler class is $e(TM) = \frac{1}{(2\pi)^n} \text{Pf}(\Omega)$, and the Pfaffian is

$$\text{Pf}(\Omega) = \frac{1}{2^n n!} \epsilon^{a_1 a_2 \dots a_{2n}} \Omega_{a_1 a_2} \wedge \dots \wedge \Omega_{a_{2n-1} a_{2n}},$$

so the normalisation constant of $\chi(M) = \int_M e(TM)$ is

$$(2\pi)^n \times (2^n n!).$$

Numerical check (carried out here for $n = 1, 2, 3, 4$):

n	$(2\pi)^n$	$2^n n!$	Product	$(4\pi)^n n!$
1	2π	2	4π	4π
2	$4\pi^2$	8	$32\pi^2$	$32\pi^2$
3	$8\pi^3$	48	$384\pi^3$	$384\pi^3$
4	$16\pi^4$	384	$6144\pi^4$	$6144\pi^4$

They agree at every n . The number is the same. There are two ways of cutting it.

3. $2^n n!$ Is One Block

The second edition's cut	$(4\pi)^n \times n!$
The natural cut	$(2\pi)^n \times (2^n n!)$

Observation 3.1 (It contradicts the second edition's own finding)

The second edition identified it correctly — “ $n!$ is the factor appearing in the definition of the Pfaffian, corresponding to the number of ways of ordering the pairs when $2n$ indices are split into n antisymmetric pairs”. But 2^n comes from the same place. The Pfaffian's denominator is the single number $2^n n!$, and the whole of it is the antisymmetrisation count: $n!$ counts the orderings of the pairs, 2^n the swap inside each pair. One cannot assign $n!$ to combinatorics while handing 2^n to geometry. The quantity $(4\pi)^n = (2\pi)^n \times 2^n$ is made by tearing the 2^n out of the Pfaffian block and gluing it to the normalisation. This is a shape the series itself has forbidden — Papers 13 and 14 laid down that a module block must not be chopped into “ $\pi \times$ a number”. What happens here is the same operation in the other direction: borrowing a factor from one block to build a new one.

4. There Are Two Standings

Factor	Value ($n = 2$)	Standing
$(2\pi)^n$	$4\pi^2$	Convention (making the periods of characteristic classes integral)
$2^n n!$	8	Combinatorics (the Pfaffian's antisymmetrisation factor)

Observation 4.1 (The 4π of $(4\pi)^n$ is not a solid angle)

$(2\pi)^n$ comes from **the convention of entering the curvature two-form as $\Omega/2\pi$** — a way of writing chosen so that the periods come out integral. It is **not the geometry itself but the scale on which the geometry is read as integers**. Hence the 4π appearing in $(4\pi)^n$ is a convention's 2π times a combinatorial 2, and is not a solid angle. This is the same shape Papers 45 and 46 found inside themselves — in 45 an area was called a solid angle, in 46 the square root of a loop factor. Here it is the product of a convention and a combinatorial count.

5. The Two-Dimensional Check of Section 2 (Unchanged)

Established Fact 5.1 (Consistency in two dimensions)

At $n = 1$ the theorem gives $\chi = \frac{1}{4\pi} \int_M R \sqrt{g} d^2x$. From classical Gauss--Bonnet $\int_M K dA = 2\pi\chi$ and the two-dimensional relation $R = 2K$,

$$\frac{1}{4\pi} \int R dA = \frac{1}{4\pi} \cdot 2 \int K dA = \frac{1}{4\pi} \cdot 2 \cdot 2\pi\chi = \chi$$

(re-verified here).

The low-dimensional values are all unchanged — 4π in two dimensions, $32\pi^2$ in four, $384\pi^3$ in six.

6. The Three Origins of the Factor 2 (The Second Edition's Judgement Is Kept)

Established Fact 6.1 (Section 4's analysis is correct)

Topological invariant	Normalisation	Origin of the factor 2
Euler characteristic χ	$\frac{1}{32\pi^2}$	Combinatorics (Pfaffian antisymmetrisation)
Instanton number k	$\frac{1}{16\pi^2}$	Component expansion (writing out $F \wedge F$; Paper 45)
θ term	$\frac{g^2}{32\pi^2}$	Generator normalisation ($\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$)

The three factors of 2 have three different origins.

This analysis is correct and is kept unchanged in the third edition. The discipline the series wrote in Paper 86 — “a common ancestor is not a common root” — and in Paper 100 — “a resonance is a signature, not a proof” — is working correctly here.

Observation 6.1 (Only the name of the substrate is corrected)

The second edition concluded: “what they share is not the factor of 2 but the substrate $(4\pi)^n$ ”. The first half is right. The second is corrected. As shown above, $(4\pi)^n$ is a quantity regrouped across a convention and a combinatorial count, and cannot serve as a substrate. What the three share is the (2π) of the characteristic-class normalisation. The $\frac{1}{(2\pi)^n}$ of the Euler class, the normalisation of the second Chern class, the way the θ term is written — each carries a 2π serving the same purpose, making the periods integral. On top of that sit factors of different origins. This is a correction, not a retraction. The second edition's intuition that there is a shared substrate was right; the name of that substrate lay one level deeper.

7. The Scope, Restated

Item	What it is
Normalisation = $(2\pi)^n(2^n n!)$	Fact (checked for $n = 1, 2, 3, 4$)
4π in 2D, $32\pi^2$ in 4D, $384\pi^3$ in 6D	Fact (values unchanged)
Two-dimensional consistency with classical GB	Fact (via $R = 2K$; re-verified)
That the factor 2 of $32\pi^2$ is $n!$	Settled (second edition; unchanged)
That $2^n n!$ is one block	The definition of the Pfaffian
$(2\pi)^n$	Convention (making the periods integral)
The grouping $(4\pi)^n$	Retracted (formed across the block)
That its 4π is a solid angle	Retracted (convention $\times$ combinatorics; Papers 45, 46)
That the three factors of 2 have different origins	Settled (second edition; unchanged)
That the shared substrate is $(4\pi)^n$	Corrected (it is (2π))
Double buffering, $SU(2)$ dimension, double cover	Retracted (second edition; not reinstated)

8. Conclusion

The second edition chased the factor of **2** in $32\pi^2$ and arrived at the $n!$ of the Pfaffian. That was right.

But another factor of the same origin was sitting beside it. The Pfaffian's denominator is $2^n n!$: $n!$ counts the orderings of the pairs, 2^n the swap inside each pair. **It is one count.**

The second edition took the 2^n out of it and glued it to the normalisation $(2\pi)^n$ to make $(4\pi)^n$. **The numbers agree. But the 4π was never a quantity that was there.**

Having tracked down where the factor of **2** came from,
it did not notice that the 2^n beside it came from the same place.

What is not claimed (restraint): 2^n is not separated from $n!$ (the same Pfaffian denominator; the shape of Papers 13, 14) / **the 4π of $(4\pi)^n$ is not called a solid angle** (convention $\times$ combinatorics; Papers 45, 46) / **the three origins of the factor 2 are not merged** (the second edition's judgement is kept; Papers 86, 100) / **the first edition's double buffering and Paper 32's $SU(2)$ dimension / double cover are not reinstated** (retracted in the second edition) / **no new theorem about the Euler characteristic is asserted** / **nothing is read into the Gauss--Bonnet term beyond the standard fact that it does not contribute to the field equations in four dimensions** / **the cosmic-OS lens is not placed in the body** (Layer 3, quarantined) / **no new mathematics and no prediction** (Layer 2).

Revision Record

Third edition (August 2026): neither the values nor the settlement of the factor 2 is altered — 4π in two dimensions, $32\pi^2$ in four, $384\pi^3$ in six, and the second edition's settlement that the factor of 2 in $32\pi^2$ is the $n!$ of the Pfaffian's antisymmetrisation all stand. The first edition's “double buffering” and Paper 32's “ $SU(2)$ dimension / double cover” remain retracted. (i) The seam of the normalisation constant was moved to the right place: since $e(TM) = \frac{1}{(2\pi)^n} \text{Pf}(\Omega)$ and $\text{Pf}(\Omega) = \frac{1}{2^n n!} \epsilon \Omega \cdots \Omega$, the normalisation constant is $(2\pi)^n \times (2^n n!)$ (checked for $n = 1, 2, 3, 4$). (ii) It was made explicit that $2^n n!$ is one block — $n!$ counting the orderings of the pairs and 2^n the swap inside each pair, **both being the same denominator of the Pfaffian**. The second edition assigned $n!$ to combinatorics while handing 2^n to the normalisation, **which does not agree with its own finding**. (iii) The grouping $(4\pi)^n$ was retracted: it is a quantity formed across a convention $((2\pi)^n)$ and a combinatorial count (2^n) , and the 4π appearing in it is **not a solid angle** (the same shape as Paper 45, which called an area a solid angle, and Paper 46, the square root of a loop factor). (iv) Section 4's “**what they share is the substrate $(4\pi)^n$** ” was corrected to “**what they share is the (2π) of the characteristic-class normalisation**”. The judgement that the three factors of 2 have three different origins is itself **correct and is unchanged**. (v) The cosmic-OS lens was removed from the body (Layer 3).

References

1. Chern, S.-S. (1944). *A Simple Intrinsic Proof of the Gauss--Bonnet Formula for Closed Riemannian Manifolds*. Annals of Mathematics 45, 747--752.
2. Chern, S.-S. (1945). *On the Curvatura Integra in a Riemannian Manifold*. Annals of Mathematics 46, 674--684.
3. Eguchi, T., Gilkey, P. B. & Hanson, A. J. (1980). *Gravitation, Gauge Theories and Differential Geometry*. Physics Reports 66, 213--393.
4. Nakahara, M. (2003). *Geometry, Topology and Physics*, 2nd ed. IOP Publishing. (the Euler class and the Pfaffian)
5. Yamagishi, Y. (2026). *An Area Is Not a Solid Angle* [Paper 45], 3rd ed. Zenodo. <https://doi.org/10.5281/zenodo.21602337>
6. Yamagishi, Y. (2026). *Of $16\pi^2$, Only $\Omega_3 = 2\pi^2$ Is Geometry* [Paper 46], 3rd ed. Zenodo. [10.5281/zenodo.21602422](https://doi.org/10.5281/zenodo.21602422)
7. Yamagishi, Y. (2026). *The Power of 4π Is the Mass Dimension* [Paper 44], 3rd ed. Zenodo. <https://doi.org/10.5281/zenodo.21602231>

8. Yamagishi, Y. (2026). *Convention and Fact* [Paper 95]. Zenodo. <https://doi.org/10.5281/zenodo.21758621>
9. Yamagishi, Y. (2026). *The Four Roots of Necessity* [Paper 86]. Zenodo. <https://doi.org/10.5281/zenodo.21744839>
10. Yamagishi, Y. (2026). *The Crossing of Royal Roads* [Paper 100]. Zenodo. <https://doi.org/10.5281/zenodo.21758864>
11. Yamagishi, Y. (2026). *The Yamagishi Gravitational Constant Unpacking Theory* [Paper 35], 3rd ed. Zenodo. <https://doi.org/10.5281/zenodo.21430118>
12. Yamagishi, Y. (2026). *The Two Patches Are Not the Same Kind of Thing* [Paper 14], 3rd ed. Zenodo. <https://doi.org/10.5281/zenodo.21600268>

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)