

# The Same Words “Critical Dimension” Name Two Different Numbers, Above and Below

— Above it is **4**, where fluctuations stop breaking mean field — below it is **2**, where  $\int_{\epsilon} k^{d-3} dk$  stops diverging  
— measured,  $d = 2$  diverges logarithmically (dividing  $\epsilon$  by  $10^4$  adds **9.210340** each time) while  $d = 3$   
converges to  $1 - \epsilon$  — the boundary sits exactly at  $d = 2$ , by a different mechanism from the **4** above — [Paper  
482]

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— *From thinking to syncing.* —

## Abstract

The critical dimension is taken to be one number. This paper shows that there are two, above and below, with different mechanisms—above is **4** and concerns the ultraviolet, below is **2** and concerns the infrared. No new theorem or law is claimed. First,  $\int_{\epsilon}^1 k^{d-3} dk$  at  $\epsilon = 10^{-6}$  gives **999999.000000** for  $d = 1$ , **13.815511** for  $d = 2$ , **0.999999** for  $d = 3$  and **0.500000** for  $d = 4$  (Section 2). Second, the  $d = 3$  value is exactly  $1 - \epsilon$  and tends to 1 as  $\epsilon$  falls (Section 2). Third, this is the core of the paper. The boundary sits exactly at  $d = 2$ — $d = 2.01$  converges at **24.142242** while  $d = 1.99$  lies on the divergent side at **31.8257** (Section 2). Fourth, the divergence at  $d = 2$  is logarithmic—each division of  $\epsilon$  by  $10^4$  adds the same **9.210340** (Section 3). Fifth, the Bose gas has the same boundary—its excited-state count diverges at **13.586173** for  $d = 2$  and settles at **0.849847** for  $d = 3$  (Section 3). Sixth, the separator is whether the cut is ultraviolet or infrared (Section 4). **Keywords:** critical dimension, upper critical dimension, lower critical dimension, phase transition, Mermin-Wagner theorem.

Scope of this paper (scope note): No new theorem or law is claimed—upper and lower critical dimensions are both known, and the dimensional dependence of infrared divergence is textbook material. Condensation itself is not discussed—existence proofs for the transition, finite-size behaviour and the interacting case are not the subject; only the infrared side of the ideal gas. Five fields were not computed—other settings share the same integrand, but numbers are given here for only two: the general integral and the Bose gas. The rest are named and left there.  $d$  is treated as a real number— $d = 1.99$  and  $d = 2.01$  are not physical dimensions but an operation for bracketing the boundary at exactly 2. Relation to earlier papers: Paper 279 showed that the cost of self-avoidance vanishes above four dimensions—the 4 there is what this paper calls “above”. Paper 271 showed that mean field does not see the dimension—the same 4. Paper 94 showed that “dimension” is not one thing—this paper splits not that word but the compound “critical dimension”. Paper 481 showed that without separators a refutation has no destination—this paper writes one of them. What is added is evaluating  $\int_{\epsilon}^1 k^{d-3} dk$  at  $d = 1, 2, 3, 4$ , confirming that  $d = 3$  gives exactly  $1 - \epsilon$ , bracketing the boundary with  $d = 2.01$  and  $d = 1.99$ , giving the logarithmic signature as an increment of 9.210340, and confirming the same boundary in the Bose gas.

## 1. Notation and the Question

Setting: In a  $d$ -dimensional wavenumber space the density of states goes as  $k^{d-1}$ . Summing a quantity that behaves as  $1/k^2$  at long wavelength gives the infrared integral

$$\int_{\epsilon}^1 k^{d-1} \cdot \frac{1}{k^2} dk = \int_{\epsilon}^1 k^{d-3} dk$$

where  $\epsilon$  is the lower cutoff set by the size of the system. The question is whether this stays finite or diverges as  $\epsilon \rightarrow 0$ .

### Observation 1.1

The critical dimension is taken to be one number. Above some dimension things get simple—a picture of a single boundary.

### Observation 1.2

But the corpus has already produced a 4. Paper 279 showed that the cost of self-avoidance vanishes above four dimensions—what follows is a different number.

Question: Where is the lower boundary, and how does it differ from the 4 above.

## 2. The Boundary Sits Exactly at Two

Computation 2.1: Evaluate  $\int_{\varepsilon}^1 k^{d-3} dk$  numerically at  $\varepsilon = 10^{-6}$ .

| Dimension $d$ | Value         | Behaviour              |
|---------------|---------------|------------------------|
| 1             | 999999.000000 | power divergence       |
| 2             | 13.815511     | logarithmic divergence |
| 3             | 0.999999      | convergent             |
| 4             | 0.500000      | convergent             |

**Observation 2.2**

The  $d = 3$  value is exactly  $1 - \varepsilon$ .0.9999990000000000 against 0.9999990000000000—the integrand becomes  $k^0 = 1$ , and agreement with the closed form is an independent check.

**Observation 2.3**

Lowering  $\varepsilon$  takes  $d = 3$  towards 1.0.999999999999—the convergent side.

**Observation 2.4**

$d = 4$  settles at exactly  $1/2$ .0.5000000000—the value of  $\int_0^1 k dk$ , again matching the closed form.

**Observation 2.5**

Taking  $\varepsilon$  to  $10^{-12}$  doubles the  $d = 2$  value.A factor of 2.0000000000—the signature of a logarithmic divergence: double the decades, double the value.

Observation 2.6

The same operation multiplies  $d = 1$  by  $10^6.1.000001 \times 10^6$ —a power divergence, on a different timescale from the logarithm.

Observation 2.7

The boundary sits exactly at  $d = 2$ .  $d = 2.01$  converges at 24.142242 and  $d = 1.99$  lies on the divergent side at 31.8257—both sides of the boundary were taken.

3. The Bose Gas Has the Same Boundary

Computation 3.1: Count the excited states of an ideal Bose gas on the infrared side alone. The integrand is  $k^{d-1}/(e^{\beta k^2} - 1)$ , which matches  $k^{d-3}$  as  $k \rightarrow 0$ . At  $\beta = 1$  and  $k = 10^{-3}$  the ratio is 0.999999500.

| Dimension $d$ | Excited count (infrared) | Verdict                   |
|---------------|--------------------------|---------------------------|
| 2             | 13.586173                | divergent (no condensate) |
| 3             | 0.849847                 | finite (condensate)       |

Observation 3.2

The  $d = 2$  value is the larger.  $13.586173 > 0.849847$ —the excited states can absorb arbitrarily many particles, so none pile into the ground state.

Computation 3.3: Tabulate the  $d = 2$  value as  $\epsilon$  falls.

| $\epsilon$ | Excited count | Increment |
|------------|---------------|-----------|
| $10^{-4}$  | 8.981003      | —         |
| $10^{-8}$  | 18.191343     | 9.210340  |
| $10^{-12}$ | 27.401684     | 9.210340  |

#### Observation 3.4

The increments agree. Their ratio is 1.000000—arithmetic growth is the signature of a logarithmic divergence.

#### Observation 3.5

The increment matches that of  $-\ln \varepsilon$ . 9.210340 against 9.210340—that is  $\ln(10^4)$ , computed independently.

#### Observation 3.6

$d = 3$  barely moves under the same operation. A ratio of 1.000117682—the convergent side: enlarging the system does not raise the excited count without limit.

### 4. The Separator

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The separator: is the cut ultraviolet or infrared.

#### Observation 4.1

The 4 above is an ultraviolet matter. Whether short-wavelength fluctuations shift the mean-field answer—they stop doing so for  $d > 4$ , and Paper 271's “mean field does not see the dimension” stands there.

#### Observation 4.2

The 2 below is an infrared matter. Whether long-wavelength fluctuations make the integral diverge—they do for  $d \leq 2$ , and order fails to form.

#### Observation 4.3

One phrase names two numbers. Both are called the critical dimension—distinct roots by the criterion of Paper 300, differing in mechanism, in which end is cut, and in the number itself.

#### Observation 4.4

There is an operation that tells them apart. Lower  $\varepsilon$  and see divergence, and it is the lower one; lower the lattice spacing instead, and it is the upper—which knob was turned decides it.

#### Observation 4.5

This is one of the separators of Paper 481. When “behaviour changes at the critical dimension” is refuted, the blame has two destinations—recording which boundary was meant removes one.

#### Observation 4.6

Paper 94 split the word “dimension” itself. What is split here is the compound “critical dimension”—splitting a word does not split the words that contain it.

#### Observation 4.7

To be honest—numbers are given here for two cases only. The general integral and the Bose gas—other settings where the same  $k^{d-3}$  appears are named but not computed. What was counted is the location of the boundary, not the physics of each field.

## 5. Roll Call and Closing

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1. 999999.000000 at  $d = 1$  (Computation 2.1): **power divergence.**
2.  $d = 3$  is exactly  $1 - \varepsilon$  (Observation 2.2): **matching the closed form.**
3.  $d = 4$  is exactly  $1/2$  (Observation 2.4): **an independent check.**
4.  $d = 2$  doubles as  $\varepsilon$  falls (Observation 2.5): **logarithmic.**
5. The boundary is exactly  $d = 2$  (Observation 2.7): **the core.**
6. Bose gives 13.586173 at  $d = 2$  (Computation 3.1): **the non-condensing side.**
7. Increments agree at 9.210340 (Observation 3.5): **that is  $\ln(10^4)$ .**
8.  $d = 3$  moves by 1.000117682 (Observation 3.6): **it saturates.**
9. The separator is ultraviolet or infrared (Observation 4.3): **distinct roots by Paper 300.**

10. Two cases were computed (Observation 4.7): the rest are only named.

Closing: The critical dimension is taken to be one number—a picture of a single boundary above which things get simple. But the corpus has already produced a 4: Paper 279 showed that the cost of self-avoidance vanishes above four dimensions. What follows is a different number. The boundary sits exactly at two—at  $\varepsilon = 10^{-6}$ ,  $\int_{\varepsilon}^1 k^{d-3} dk$  gives **999999.000000** for  $d = 1$ , **13.815511** for  $d = 2$ , **0.999999** for  $d = 3$  and **0.500000** for  $d = 4$ . The  $d = 3$  value is exactly  $1 - \varepsilon$ , since the integrand becomes  $k^0 = 1$ , and that agreement with the closed form is an independent check; lowering  $\varepsilon$  takes it to 0.999999999999.  $d = 4$  settles at exactly  $1/2$ , the value of  $\int_0^1 k dk$ . Taking  $\varepsilon$  to  $10^{-12}$  doubles the  $d = 2$  value—the signature of a logarithmic divergence, where doubling the decades doubles the value—while the same operation multiplies  $d = 1$  by  $10^6$ . And the boundary is exactly  $d = 2$ :  $d = 2.01$  converges at **24.142242**,  $d = 1.99$  diverges at **31.8257**. The Bose gas has the same boundary—counting the excited states of an ideal Bose gas on the infrared side gives **13.586173** at  $d = 2$ , divergent, and **0.849847** at  $d = 3$ , finite. In the first the excited states absorb arbitrarily many particles, so none pile into the ground state. Lowering  $\varepsilon$  from  $10^{-4}$  to  $10^{-8}$  to  $10^{-12}$  gives 8.981003, 18.191343 and 27.401684, with increments agreeing at **9.210340**. That value matches the increment of  $-\ln \varepsilon$ , namely  $\ln(10^4)$ .  $d = 3$  moves by a ratio of only 1.000117682: enlarging the system does not raise the count without limit. One thing separates them—is the cut ultraviolet or infrared. The 4 above is ultraviolet: whether short-wavelength fluctuations shift the mean-field answer, which they stop doing for  $d > 4$ , where Paper 271’s “mean field does not see the dimension” stands. The 2 below is infrared: whether long-wavelength fluctuations make the integral diverge, which they do for  $d \leq 2$ , and order fails to form. Both are called the critical dimension, yet they are distinct roots by the criterion of Paper 300, differing in mechanism, in which end is cut, and in the number itself. An operation tells them apart—lower  $\varepsilon$  and see divergence and it is the lower one, lower the lattice spacing instead and it is the upper. This is one of the separators of Paper 481: when “behaviour changes at the critical dimension” is refuted, the blame has two destinations, and recording which boundary was meant removes one. Paper 94 split the word “dimension” itself; what is split here is the compound “critical dimension”—splitting a word does not split the words that contain it. To be honest—numbers are given here for the general integral and the Bose gas only. Other settings where the same  $k^{d-3}$  appears are named but not computed. What was counted is the location of the boundary, not the physics of each field.

## References

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1. Mermin, N. D., & Wagner, H. (1966). Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models. *Physical Review Letters*.
  2. Hohenberg, P. C. (1967). Existence of Long-Range Order in One and Two Dimensions. *Physical Review*.
  3. Ginzburg, V. L. (1960). Some Remarks on Phase Transitions of the Second Kind and the Microscopic Theory of Ferroelectric Materials. *Soviet Physics Solid State*.
  4. Yamagishi, Y. (2026). “Dimension” Is Not One Thing [Paper 94]. Zenodo. 10.5281/zenodo.21758581

5. Yamagishi, Y. (2026). *Mean Field Does Not See the Dimension, and Yet the Dimension Alone Settles How Far It Errs* [Paper 271]. Zenodo. 10.5281/zenodo.22169925
6. Yamagishi, Y. (2026). *What Changed the Exponent of a Chain Was Self-Avoidance Alone, and above Four Dimensions That Cost Disappears* [Paper 279]. Zenodo. 10.5281/zenodo.22170222
7. Yamagishi, Y. (2026). *The Three Hundredth Census: Whether Two Things Share a Root Is Decidable* [Paper 300]. Zenodo. 10.5281/zenodo.22171914
8. Yamagishi, Y. (2026). *A Broken Law Can Be Blamed on Any of the Assumptions* [Paper 481]. Zenodo. 10.5281/zenodo.22733927

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