

A Stationary Point Comes in Three Kinds: Minimum, Maximum, and Everywhere Equal

— At a flat mirror the path length **2.828427125** is shorter than both neighbours, a minimum; at a strongly curved mirror the vertex **3.572114220** is longer than both neighbours at **3.571008542**, a maximum — between the foci of an ellipse all **24** sampled points give **4.000000000000** with a spread of **4.441×10^{-16}** , neither minimum nor maximum — the same stationarity yields Snell's law, **0.806200914** against **0.806200883** —

[Paper 487]

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— *From thinking to syncing.* —

Abstract

Light is said to take the shortest path. This paper shows that a stationary point comes in three kinds — minimum, maximum and degenerate — and that all three can be built in optics—at an elliptical mirror all 24 points agree to within **4.441×10^{-16}** . No new theorem or law is claimed. First, at a flat mirror the stationary point sits at $x = 0.000000017$ and the path length **2.828427125** is shorter than both neighbours at 2.831969281—a minimum (Section 2). Second, the angles of incidence and reflection agree there, **45.000000** degrees against **45.000000** (Section 2). Third, this is the core of the paper. At a strongly curved mirror the vertex value **3.572114220** exceeds both neighbours at **3.571008542**—the same stationarity gives a maximum (Section 3). Fourth, reflecting between the foci of an ellipse, all 24 sampled points give a path length of **4.000000000000**—a spread of **4.441×10^{-16}** , neither minimum nor maximum (Section 3). Fifth, the same stationarity yields Snell's law at refraction— **$n_1 \sin \theta_1 = 0.806200914$** against **$n_2 \sin \theta_2 = 0.806200883$** (Section 4). Sixth, the separator is whether stationarity or minimality is required (Section 4). **Keywords:** Fermat's principle, stationary point, variational principle, optical path length, extremum.

Scope of this paper (scope note): No new theorem or law is claimed—that Fermat’s principle is stationarity rather than minimality is standard, and Paper 84 already states in words that the action is stationary and not necessarily minimal, saddle points included. What this paper adds is numbers—minimum, maximum and degenerate are each built in optics and their values set side by side. The degenerate case in particular is not covered by the phrase “saddle point”. No general variational theory—second variations, conjugate points and Morse index theory are not entered; only what a one-variable numerical search reaches. Wave optics is excluded—diffraction, interference and the stationary-phase account through path integrals are not the subject; only geometrical path length. No teleology—as Papers 84 and 97 established, light is not written as “choosing” a path. Stationarity is a local condition. Relation to earlier papers: Paper 84 showed that least action is stationarity and not teleology—this paper attaches numbers, and a counterexample, to that sentence. Paper 143 showed that minimum entropy production is not a principle but a theorem with a fence—the same shape: a claim named for a minimum needing a fence. Paper 486 showed that optima in three fields are one vertex—there the vertex was of one kind; here the vertex has three. Paper 481 showed that without separators a refutation has no destination—this paper writes one of them. What is added is the minimum as 2.828427125 against 2.831969281, the maximum as 3.572114220 against 3.571008542, the degenerate case as a spread of 4.441×10^{-16} , setting the three in one frame, and confirming Snell’s law numerically from the stationarity condition.

1. Notation and the Question

Setting: Write the optical path length from A via a mirror point x to B as

$$L(x) = n_1 |Ax| + n_2 |xB|$$

Fermat’s principle requires $L'(x) = 0$, that is, that L be stationary. It does not require that L be minimal. Below, the shape of the mirror is varied and L examined.

Observation 1.1

Light is said to take the shortest path. The minimal route—the picture that travels with “nature wastes nothing”.

Observation 1.2

But the condition is $L'(x) = 0$. A first-order condition—saying nothing about the sign of the second.

Question: Can a stationary point that is not a minimum actually be built.

2. The Case That Is a Minimum

Computation 2.1: Searching for the stationary point of $L(x)$ at a flat mirror gives $x = 0.000000017$ with a path length of **2.828427125**. It lies inside the search interval $[-5, 5]$ and is not pinned to an endpoint.

Observation 2.2

It is a minimum. $2.828427125 < 2.831969281$ —shorter than both neighbours: a valley.

Observation 2.3

The stationary point sits at the symmetric position. $x = 0.000000017$ —exactly 0, with the remainder being the step size of the search.

Observation 2.4

The angles of incidence and reflection agree there. 45.000000 degrees against 45.000000 —the law of reflection falling out of stationarity, an independent check.

Observation 2.5

Up to here “the shortest path” causes no trouble. A flat mirror gives a minimum—on this example alone, calling it a minimum principle raises no contradiction.

3. The Case That Is a Maximum, and the Case Where Everything Is Equal

Computation 3.1: Increase the curvature of the mirror and repeat the search.

Position	Path length
Vertex	3.572114220
Left neighbour	3.571008542
Right neighbour	3.571008542

Observation 3.2

The stationary point is a maximum. $3.572114220 > 3.571008542$ —longer than both neighbours: a peak. The same $L'(x) = 0$ pointed at a hill, not a valley.

Observation 3.3

So it is stationarity, not minimality.Both a minimum and a maximum exist—one counterexample is enough to unseat the word “minimum”.

Computation 3.4: Place A and B at the foci of an ellipse and reflect at 24 points on it.

Quantity	Value
Path length (all 24 points)	4.00000000000000
Spread over 24 points	4.441×10^{-16}
Major axis $2a$	4.0000

Observation 3.5

Every reflection point gives the same length.A spread of 4.441×10^{-16} —the size of double-precision rounding, with all 24 points equal.

Observation 3.6

The value matches the major axis. 4.000000000000 against $2a = 4.0000$ —the value the definition of the ellipse gives, an independent check.

Observation 3.7

This is neither a minimum nor a maximum. Every point is stationary—neither valley nor hill but flat. The phrase “saddle points included” does not reach this case.

4. The Separator

The separator: is what is required stationarity, or minimality.

Observation 4.1

Stationarity, and there are three kinds. Minimum, maximum, degenerate—all three were built in optics.

Observation 4.2

Minimality, and the curved mirror is a counterexample. $3.572114220 > 3.571008542$ —it satisfies the first-order condition and is not a minimum.

Observation 4.3

The same stationarity also yields the law of refraction. At the stationary point $x = 0.362650234$, $n_1 \sin \theta_1 = 0.806200914$ and $n_2 \sin \theta_2 = 0.806200883$ —Snell’s law, whose side does not depend on minimum or maximum.

Observation 4.4

So it is the first-order condition alone that carries the physics. The sign of the second is set by the shape—the curvature of the mirror, a quantity outside the principle.

Observation 4.5

Paper 84 said the same thing in words. Stationary and not necessarily minimal—what is added here is numbers, and the third case. “Saddle points included” drops the degenerate one.

Observation 4.6

It has the shape of Paper 143 as well. Minimum entropy production was not a principle but a theorem with a fence—claims named for a minimum usually need one.

Observation 4.7

To be honest—the degenerate case depends on a special arrangement. It occurs only at the two foci—move off a focus and the flatness is gone, so of the three kinds only the degenerate one is unstable. The three were set out as equals, but they are not met equally often.

5. Roll Call and Closing

1. **2.828427125** at a flat mirror (Computation 2.1): **shorter than both neighbours.**
2. **45.000000** degrees each way (Observation 2.4): **an independent check.**
3. **3.572114220** at the vertex (Computation 3.1): **the core.**
4. Neighbours at **3.571008542** (Observation 3.2): **the stationary point is a maximum.**
5. **4.000000000000** at all **24** points (Computation 3.4): **a spread of 4.441×10^{-16} .**
6. **Matching the major axis $2a$** (Observation 3.6): **from the definition.**
7. **Degenerate is not a saddle** (Observation 3.7): **neither valley nor hill.**
8. **Snell’s law follows** (Observation 4.3): **0.806200914** against **0.806200883.**
9. **The separator is stationary or minimal** (Observation 4.1): **stationary gives three.**

10. The degenerate case is unstable (Observation 4.7): only at the foci.

Closing: Light is said to take the shortest path—the minimal route, the picture that travels with “nature wastes nothing”. But the condition is $L'(x) = 0$, a first-order condition that says nothing about the sign of the second. The case that is a minimum—at a flat mirror the stationary point sits at $x = 0.000000017$ with path length **2.828427125**, shorter than both neighbours at 2.831969281, and lies inside the search interval $[-5, 5]$ rather than pinned to an endpoint. It sits at the symmetric position, exactly 0, the remainder being the step size of the search. The angles of incidence and reflection agree there at 45.000000 degrees, so the law of reflection falls out of stationarity as an independent check. Up to here “the shortest path” causes no trouble, and on this example alone calling it a minimum principle raises no contradiction. The case that is a maximum, and the case where everything is equal—increase the curvature and the vertex value **3.572114220** exceeds both neighbours at 3.571008542: the same $L'(x) = 0$ pointed at a hill, not a valley. With both a minimum and a maximum in hand, it is stationarity and not minimality. Reflecting between the foci of an ellipse, all 24 sampled points give **4.000000000000** with a spread of 4.441×10^{-16} , the size of double-precision rounding. The value matches the major axis $2a = 4.0000$, which the definition of the ellipse gives, an independent check. This is neither minimum nor maximum: every point is stationary, neither valley nor hill but flat, and the phrase “saddle points included” does not reach it. One thing separates them—is what is required stationarity, or minimality. Stationarity, and there are three kinds—minimum, maximum, degenerate—all three built here in optics. Minimality, and the curved mirror is a counterexample satisfying the first-order condition without being a minimum. The same stationarity yields the law of refraction: at $x = 0.362650234$, $n_1 \sin \theta_1 = 0.806200914$ and $n_2 \sin \theta_2 = 0.806200883$, Snell’s law, whose side does not depend on minimum or maximum. So the first-order condition alone carries the physics, while the sign of the second is set by the curvature of the mirror, a quantity outside the principle. Paper 84 said the same thing in words—stationary and not necessarily minimal—and what is added here is numbers and the third case, since “saddle points included” drops the degenerate one. It has the shape of Paper 143 as well: minimum entropy production was not a principle but a theorem with a fence, and claims named for a minimum usually need one. To be honest—the degenerate case depends on a special arrangement. It occurs only at the two foci, and moving off a focus removes the flatness, so of the three kinds only the degenerate one is unstable. The three were set out as equals, but they are not met equally often.

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