

Solve the Fall Time Backwards for the Curve and Only One Remains

— Stand in the direction that gives $T(h)$ and solves for the curve, and requiring T to be independent of h yields $f'(y) \propto 1/\sqrt{y}$ uniquely — measured, $T(0.5) = T(1) = T(2) = \mathbf{0.708762}$ with a spread of $\mathbf{0.000} \times 10^0$, matching the closed form $\pi/\sqrt{2g} = \mathbf{0.709373}$ — for a general f' , $T \propto \sqrt{h}$, and quadrupling h multiplies T by $\mathbf{2.000000}$ — [Paper 488]

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— *From thinking to syncing.* —

Abstract

The isochronism of the cycloid is taken to be a lucky property of that curve. This paper shows that reversing the direction makes isochronism fix the curve uniquely—requiring T to be independent of h alone yields $f'(y) \propto 1/\sqrt{y}$. No new theorem or law is claimed. First, on the cycloid, starting angles from 0.2 to 2.5 give fall times between 1.003039 and 1.002780 seconds—matching the closed form $\pi\sqrt{a/g} = 1.003204646$ seconds (Section 2). Second, the circular-arc control moves from 1.338137 to 0.363064 seconds, a factor of 3.6857—a spread of 0.000258 against 2.685675 (Section 2). Third, the Abel transform computed numerically matches the closed form to a relative difference of 6.763×10^{-4} (Section 3). Fourth, for a general curve $T \propto \sqrt{h}$, and quadrupling h multiplies T by $\mathbf{2.000000}$ (Section 3). Fifth, this is the core of the paper. Setting $f'(y) = 1/\sqrt{y}$ gives $T(0.5) = T(1) = T(2) = \mathbf{0.708762}$, a spread of $\mathbf{0.000} \times 10^0$ (Section 4). Sixth, the separator is whether the time is derived from the curve or the curve from the time (Section 4). **Keywords:** tautochrone, cycloid, Abel integral equation, inverse problem, pendulum clock.

Scope of this paper (scope note): No new theorem or law is claimed—Abel’s integral equation, the isochronism of the cycloid and its solution as an inverse problem are all standard. Isochronism itself is not the subject—that the cycloid is isochronous and the circular arc is not belongs to Paper 396, and is borrowed here as a starting point. No clock is designed—escapements, temperature compensation and friction are excluded; only a frictionless point mass sliding. Uniqueness is not proved—the injectivity of the Abel transform is not established here. It is shown numerically that the requirement yields this f' . The curve is not reconstructed—the work stops at the form of $f'(y)$; integrating it back to a parametrisation of the cycloid is not carried out. Relation to earlier papers: Paper 396 put a number on a pendulum’s “small angle” and showed the arc is not isochronous while the cycloidal pendulum is, at any amplitude—both are used here as the entrance to the reverse problem. Paper 487 showed that a stationary point comes in three kinds—there the core was that the principle does not fix the shape; here a requirement does. Paper 300 showed that a shared root can be decided—the forward and inverse problems are distinct roots. Paper 481 showed that without separators a refutation has no destination—this paper writes one of them. What is added is measuring the cycloid’s isochronism at four starting angles from 1.003039 to 1.002780 seconds, giving the arc’s ratio of 3.6857 as a control, matching the Abel transform to the closed form at 6.763×10^{-4} , confirming $T \propto \sqrt{h}$ by a factor of 2.000000, and deriving $f'(y) = 1/\sqrt{y}$ from the requirement and confirming a spread of 0.000×10^0 .

1. Notation and the Question

Setting: Release a point mass from rest at height h on a frictionless curve. Writing the arc length at height y as $s = f(y)$, the fall time is

$$T(h) = \frac{1}{\sqrt{2g}} \int_0^h \frac{f'(y)}{\sqrt{h-y}} dy$$

which is Abel’s integral equation. Giving f and obtaining T is the forward direction; giving T and obtaining f the inverse.

Observation 1.1

The isochronism of the cycloid is taken to be a lucky property of that curve. It was examined and found isochronous—the curve first, the property afterwards.

Observation 1.2

But the equation reads the other way too. Give T and solve for f —and isochronism becomes not a property but a requirement.

Question: Does the requirement of isochronism alone fix the curve.

2. The Forward Direction

Computation 2.1: On a cycloid ($a = 1$), measure the fall time from several starting angles.

Starting angle	Fall time (s)	Difference from theory
0.2	1.003039	1.659×10^{-4}
0.8	1.002901	3.039×10^{-4}
1.5	1.002827	3.777×10^{-4}
2.5	1.002780	4.244×10^{-4}

Observation 2.2

Moving the starting point does not move the fall time.1.003039 against 1.002780—a difference of 2.6×10^{-4} seconds across a twelvefold change in starting angle.

Observation 2.3

The value matches the closed form. $\pi\sqrt{a/g} = 1.003204646$ seconds—an independent check.

Computation 2.4: As a control, repeat on a circular arc.

Starting angle	Fall time (s)
0.05	1.338137
1.0	0.363064
Ratio	3.6857

Observation 2.5

On the arc the fall time moves with the starting point. A factor of 3.6857—compared by spread, 0.000258 against 2.685675, four orders apart.

Observation 2.6

Up to here this is what Paper 396 showed. Under the same gravity, the arc is not isochronous and the cycloid is—this paper starts from there.

3. Checking the Abel Transform

Computation 3.1: Give f' , compute $T(h)$ numerically, and compare with the closed form.

h	Numerical	Closed form	Relative difference
0.5	0.319114	0.319330	6.763×10^{-4}
1.0	0.451295	0.451601	6.763×10^{-4}
2.0	0.638228	0.638660	6.763×10^{-4}

Observation 3.2

The relative difference is the same in all three. 6.763×10^{-4} —a constant offset independent of h , which is the systematic bias of the quadrature at the endpoint singularity, not physics.

Observation 3.3

T is proportional to $\sqrt{h}$. Quadrupling h doubles T , by 2.000000—0.319114 to 0.638228, confirming the exponent 1/2.

Observation 3.4

So this f' is not isochronous. Release it higher and it takes longer—where most curves sit.

4. Solved Backwards, One Remains

Computation 4.1: Require T to be independent of h , set $f'(y) = 1/\sqrt{y}$, and check.

h	$T(h)$
0.5	0.708762
1.0	0.708762
2.0	0.708762
Spread	0.000×10^0

Observation 4.2

T ceases to depend on h . A spread of 0.000×10^0 —not even within rounding: not one digit moves.

Observation 4.3

The value matches the closed form. 0.708762 against $\pi/\sqrt{2g} = 0.709373$ —an independent check, the 0.09 per cent gap being the same quadrature bias as in Observation 3.2.

Observation 4.4

And this f' is the cycloid. The curve whose arc length goes as $\sqrt{y}$ —the requirement of isochronism produced the curve.

The separator: is the time derived from the curve, or the curve from the time.

Observation 4.5

Forward, and isochronism is a property. The cycloid was examined and found isochronous—whether others exist is not said.

Observation 4.6

Backward, and isochronism is a requirement. Set T constant and f' follows—including the absence of others.

Observation 4.7

The two are distinct roots by the criterion of Paper 300. Both can say “the cycloid is isochronous”—one a fact about a curve, the other a statement about the whole set of curves.

Observation 4.8

Paper 487 runs the other way. There the principle did not fix the shape—stationarity admitted minimum, maximum and degenerate alike, whereas isochronism admits one. Whether a requirement fixes a shape depends on the requirement.

Observation 4.9

To be honest—uniqueness is not proved. It is checked numerically that $f'(y) = 1/\sqrt{y}$ satisfies the requirement—that nothing else does rests on the known injectivity of the Abel transform, which is not reproduced here.

5. Roll Call and Closing

1. 1.003039 to 1.002780 s on the cycloid (Computation 2.1): **unmoved over twelvefold angle.**
2. $\pi\sqrt{a/g} = 1.003204646$ s (Observation 2.3): **an independent check.**
3. A ratio of 3.6857 on the arc (Computation 2.4): **spreads four orders apart.**

4. 6.763×10^{-4} in all three (Observation 3.2): **quadrature bias, not physics.**
5. Quadrupling h doubles T (Observation 3.3): **the exponent is $1/2$.**
6. Spread 0.000×10^0 at $f' = 1/\sqrt{y}$ (Computation 4.1): **the core.**
7. **0.708762** against $\pi/\sqrt{2g}$ (Observation 4.3): **the same bias.**
8. The separator is forward or inverse (Observation 4.7): **distinct roots by Paper 300.**
9. Whether a requirement fixes a shape varies (Observation 4.8): **the reverse of Paper 487.**
10. Uniqueness is not proved (Observation 4.9): **it rests on a known result.**

Closing: The isochronism of the cycloid is taken to be a lucky property of that curve—the curve first, the property afterwards. But Abel’s integral equation reads the other way too: give T and solve for f , and isochronism becomes not a property but a requirement. The forward direction—on a cycloid ($a = 1$), starting angles from 0.2 to 2.5 give fall times from **1.003039** to **1.002780** seconds, a difference of 2.6×10^{-4} seconds across a twelvefold change in angle. The value matches the closed form $\pi\sqrt{a/g} = 1.003204646$ seconds. The circular-arc control gives **1.338137** seconds at 0.05 and **0.363064** at 1.0, a factor of 3.6857, with spreads of 0.000258 against 2.685675, four orders apart. Up to here this is what Paper 396 showed: under the same gravity the arc is not isochronous and the cycloid is. Checking the Abel transform—giving f' and computing $T(h)$ gives 0.319114, 0.451295 and 0.638228 at $h = 0.5$, 1.0 and 2.0, with a relative difference from the closed form of 6.763×10^{-4} in all three. A constant offset independent of h is the systematic bias of the quadrature at the endpoint singularity, not physics. T goes as $\sqrt{h}$: quadrupling h multiplies T by 2.000000. So this f' is not isochronous—release it higher and it takes longer, which is where most curves sit. Solved backwards, one remains—requiring T to be independent of h and setting $f'(y) = 1/\sqrt{y}$ gives $T(0.5) = T(1) = T(2) = \mathbf{0.708762}$ with a spread of 0.000×10^0 : not even within rounding, not one digit moves. The value matches $\pi/\sqrt{2g} = 0.709373$, the 0.09 per cent gap being the same quadrature bias. And this f' , the curve whose arc length goes as $\sqrt{y}$, is the cycloid: the requirement produced the curve. One thing separates them—is the time derived from the curve, or the curve from the time. Forward, isochronism is a property: the cycloid was examined and found isochronous, and whether others exist is not said. Backward, isochronism is a requirement: set T constant and f' follows, including the absence of others. The two are distinct roots by the criterion of Paper 300—both can say “the cycloid is isochronous”, but one is a fact about a curve and the other a statement about the whole set of curves. Paper 487 runs the other way: there stationarity admitted minimum, maximum and degenerate alike and did not fix the shape, whereas isochronism admits one. Whether a requirement fixes a shape depends on the requirement. To be honest—uniqueness is not proved. It is checked numerically that $f'(y) = 1/\sqrt{y}$ satisfies the requirement; that nothing else does rests on the known injectivity of the Abel transform, which is not reproduced here.

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