

Length Is Definable as a Limit of Polylines; Area Is Not Definable as a Limit of Polyhedra

— Triangles inscribed in a cylinder give **6.283288695** under the refinement $m = n$, converging to the correct $2\pi rh$ (quadrupling n divides the error by **15.9960**) — under $m = n^2$ they settle on a different finite value, **31.636371400**, and under $m = n^3$ they diverge — while polygons inscribed in the same circle converge to **6.283169158** however they are refined — [Paper 493]

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— *From thinking to syncing.* —

Abstract

The area of a curved thing is taken to follow from approximating it with small flat pieces. This paper shows that the area of one cylinder comes out three different ways depending on the refinement—while its length comes out one way whatever the refinement. No new theorem or law is claimed. First, under $m = n$ the area is **6.283288695**, converging to the correct $2\pi rh = 6.283185307$ —quadrupling n divides the error by **15.9960** (Section 2). Second, this is the core of the paper. Under $m = n^2$ it settles on **31.636371400**—matching the closed form $2\pi\sqrt{1 + \pi^4/4} = 31.636491765$ (Section 3). Third, the same operation divides the error only by **0.9999**—it is not approaching the correct value (Section 3). Fourth, under $m = n^3$ it diverges—quadrupling n multiplies it by **4.000229** (Section 3). Fifth, polygons inscribed in the same circle converge to **6.283169158**—length does not depend on the refinement (Section 4). Sixth, the separator is whether closeness of position suffices, or the normals must align too (Section 5). **Keywords:** Schwarz lantern, surface area, polyhedral approximation, geometric measure theory.

Scope of this paper (scope note): No new theorem or law is claimed—the Schwarz lantern, and the fact that area cannot be defined as a limit of inscribed polyhedra, are standard. No definition of area is built—Hausdorff measure, developable surfaces and geometric measure theory are not entered; the counterexample is given in numbers and left there. One surface only—only a cylinder is examined; what happens for general surfaces is not treated. No convergence proof—that the $m = n^2$ limit matches the closed form is confirmed numerically, not proved. No claim about digits— n reaches only 800, and the $m = n^2$ value differs from the closed form in the fourth digit. Relation to earlier papers: Paper 154 showed that a limit without its order is not a quantity—here it is not the order but the ratio that changes the answer. Paper 245 showed that two limits have no answer until the order is written—here two limits run together and their ratio settles the answer. Paper 488 showed a case where a requirement fixes the shape—this paper is the converse: a requirement that fixes no answer. Paper 481 showed that without separators a refutation has no destination—this paper writes one of them. What is added is placing three refinements in one table, confirming the $m = n$ convergence at order 15.9960, matching the $m = n^2$ limit against a closed form, giving the $m = n^3$ divergence as 4.000229, and setting the convergent length beside them as a control.

1. Notation and the Question

Setting: Take a cylinder of radius $r = 1$ and height $h = 1$. Divide its side into m bands vertically and n sectors circumferentially, place every vertex on the cylinder, and span with triangles. Their total area is

$$A(m, n) = 2mn r \sin \frac{\pi}{n} \sqrt{\left(\frac{h}{m}\right)^2 + \left(r \left(1 - \cos \frac{\pi}{n}\right)\right)^2}$$

and the correct lateral area is $2\pi rh = 6.283185307$. The question is where A goes as m and n grow together.

Observation 1.1

The area of a curved thing is taken to follow from approximating it with small flat pieces. Span with triangles and add—refine and it approaches the truth, runs the picture.

Observation 1.2

For length that is correct. Inscribed polygons converge to the circumference—with no need to think about how the refinement is done.

Question: Does the same hold for area.

2. Under $m = n$ It Goes to the Right Value

Computation 2.1: Set $m = n$ and increase n .

n	$A(n, n)$	Error
50	6.309539658	—
200	6.284839099	1.654×10^{-3}
800	6.283288695	1.034×10^{-4}

Observation 2.2

It approaches the correct value.6.283288695 against 6.283185307—agreeing to four digits.

Observation 2.3

The convergence is second order.Quadrupling n divides the error by 15.9960—which matches 16; this is not merely a test that one value is small.

Observation 2.4

So far the naive expectation holds.Refine and it approaches—on this refinement alone, area looks definable by approximation.

3. Under $m = n^2$ It Goes Somewhere Else

Computation 3.1: Repeat with $m = n^2$, and then with $m = n^3$.

n	With $m = n^2$	With $m = n^3$
50	31.605690398	1.548797×10^3
200	31.634565972	6.200876×10^3
800	31.636371400	2.480493×10^4

Observation 3.2

Under $m = n^2$ it settles on a different finite value. **31.636371400** against the closed form $2\pi\sqrt{1 + \pi^4/4} = 31.636491765$ —matching an independently derived expression.

Observation 3.3

That value is more than five times the correct area. **31.636492** against **6.283185**—less a failed approximation than convergence to a different quantity.

Observation 3.4

The same operation does not reduce the error. Quadrupling n gives a factor of **0.9999**—the operation that gave **15.9960** under $m = n$. It is not approaching the correct value.

Observation 3.5

Under $m = n^3$ it diverges. Quadrupling n multiplies it by **4.000229**—with no ceiling: it grows without bound.

Observation 3.6

So without specifying the refinement there is no answer. Convergent, different, divergent—one phrase, “approximate by inscribed polyhedra”, carrying three answers.

4. Length Comes Out One Way, However It Is Refined

Computation 4.1: Take the perimeter of polygons inscribed in the same circle.

n	Perimeter
50	6.279051953
800	6.283169158
Correct value $2\pi r$	6.283185307

Observation 4.2

There is no refinement to choose. One subdivision count, n —with nothing to form a ratio with, the destination is unique.

Observation 4.3

The contrast with area is large. A gap of 25.353083 between $m = n$ and $m = n^2$ —for length such a gap cannot even be constructed.

Observation 4.4

The reason lies in the orientation of the faces. A polyline aligns with the tangent, but a triangle’s normal need not align—refine the bands too finely and the faces stand obliquely to the cylinder, inflating the area.

Observation 4.5

Putting every vertex on the surface does not prevent it. Every refinement keeps the vertices on the cylinder —“close to the surface” does not mean “close in area”.

5. The Separator

The separator: does closeness of position suffice, or must the normals align too.

Observation 5.1

For length, position is enough. A polyline's direction tends to the tangent by itself—hence convergence.

Observation 5.2

For area, the orientation must be close as well. Position alone will not do—the lantern brings positions close while throwing orientations off.

Observation 5.3

The two are distinct roots by the criterion of Paper 300. Both are “the limit of inscribed figures”—but one is fixed by position and the other depends on orientation too.

Observation 5.4

Paper 154 said that a limit without its order is not a quantity. Here it is not order but ratio— m and n grow together, and what settles the answer is where m/n^2 goes. Beside “order” stands “ratio”.

Observation 5.5

It also neighbours Paper 245. There, two limits had no answer until the order was written—here the order is not enough: the ratio of the two rates must be written before there is an answer.

Observation 5.6

This is one of the separators of Paper 481. When “the numerical integration does not converge to the right area” is reported—whether the mesh is coarse or the refinement ratio is wrong sends the repair elsewhere.

Observation 5.7

To be honest—the $m = n^2$ limit differs from the closed form in the fourth digit. **31.636371400** against **31.636491765**—because n reaches only 800; the rate of that convergence was not examined. Agreement can be claimed to four digits.

6. Roll Call and Closing

1. **6.283288695** under $m = n$ (Computation 2.1): **towards the correct value.**
2. **Error divided by 15.9960** (Observation 2.3): **second order.**
3. **31.636371400** under $m = n^2$ (Computation 3.1): **the core.**
4. **Matching 31.636491765** (Observation 3.2): **an independent check.**
5. **Error factor 0.9999** (Observation 3.4): **not approaching.**
6. **4.000229** under $m = n^3$ (Observation 3.5): **divergent.**
7. **Perimeter 6.283169158** (Computation 4.1): **a unique destination.**
8. **The reason is the normals** (Observation 4.4): **position is not enough.**
9. **The separator is position or orientation** (Observation 5.3): **distinct roots by Paper 300.**
10. **Four digits only** (Observation 5.7): **n reaches 800.**

Closing: The area of a curved thing is taken to follow from approximating it with small flat pieces—span with triangles, add, refine, and it approaches the truth. For length that is correct: inscribed polygons converge to the circumference with no need to think about how the refinement is done. Under $m = n$ it goes to the right value—dividing the cylinder into m bands and n sectors and spanning with triangles gives **6.283288695** at $n = 800$ under $m = n$, agreeing with the correct $2\pi r h = 6.283185307$ to four digits. The convergence is second order: quadrupling n divides the error by **15.9960**, matching **16**, so this is not merely a test that one value is small. So far the naive expectation holds, and on this refinement alone area looks definable by approximation. Under $m = n^2$ it goes somewhere else—with $m = n^2$ the value at $n = 800$ is **31.636371400**, matching the closed form $2\pi\sqrt{1 + \pi^4/4} = 31.636491765$. It is more than five times the correct area—less a failed approximation than convergence to a different quantity. The same operation divides the error only by **0.9999**, where under $m = n$ it divided by 15.9960: it is not approaching the correct value. With $m = n^3$, quadrupling n multiplies the area by **4.000229**, growing without bound. So without specifying the refinement there is no answer, and one phrase—“approximate by inscribed polyhedra”—carries three: convergent, different, divergent. Length comes out one way, however it is refined—polygons inscribed in the same circle give **6.283169158** at $n = 800$, converging to $2\pi r = 6.283185307$. There is only one subdivision count, so with nothing to form a ratio with, the destination is

unique. For area the gap between $m = n$ and $m = n^2$ is 25.353083; for length such a gap cannot even be constructed. The reason lies in the orientation of the faces: a polyline aligns with the tangent, but a triangle's normal need not, and refining the bands too finely stands the faces obliquely to the cylinder and inflates the area. Putting every vertex on the surface does not prevent it: “close to the surface” does not mean “close in area”. One thing separates them—does closeness of position suffice, or must the normals align too. For length position is enough, since a polyline's direction tends to the tangent by itself. For area the orientation must be close as well, and the lantern brings positions close while throwing orientations off. The two are distinct roots by the criterion of Paper 300—both are “the limit of inscribed figures”, but one is fixed by position and the other depends on orientation too. Paper 154 said that a limit without its order is not a quantity; here it is not order but ratio, since m and n grow together and what settles the answer is where m/n^2 goes. Beside “order” stands “ratio”. It neighbours Paper 245 as well: there two limits had no answer until the order was written, while here the order is not enough and the ratio of the two rates must be written. This is one of the separators of Paper 481: when a numerical integration fails to converge to the right area, whether the mesh is coarse or the refinement ratio is wrong sends the repair elsewhere. To be honest—the $m = n^2$ limit differs from the closed form in the fourth digit, 31.636371400 against 31.636491765, because n reaches only 800. The rate of that convergence was not examined, and agreement can be claimed to four digits.

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