

What Separates the Mean from the Median Is Itô's Second-Order Term Itself

— At $\mu = 0.10$, $\sigma = 0.20$ and $T = 1$ the mean is **110.517091808** and the median **108.328706767**, a ratio of **1.020201340** agreeing with $\exp(\sigma^2 T/2)$ to 12 digits — the growth of the mean logarithm is **0.080000000000**, exactly $\mu - \sigma^2/2$ — doubling σ multiplies the gap by **4.122849**, and at $\sigma = 0$ it vanishes — [Paper 495]

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— *From thinking to syncing.* —

Abstract

The $-\sigma^2/2$ of Black-Scholes is sometimes read as a correction for discounting or risk aversion. This paper shows that it is Itô's second-order term itself, measurable directly as the gap between mean and median—the ratio agrees with $\exp(\sigma^2 T/2)$ to 12 digits. No new theorem or law is claimed. First, the mean is **110.517091808** and the median **108.328706767** (Section 2). Second, this is the core of the paper. Their ratio **1.020201340** agrees with $\exp(\sigma^2 T/2)$ to 12 digits (Section 2). Third, the growth of the mean logarithm is **0.080000000000**, exactly $\mu - \sigma^2/2$ (Section 2). Fourth, the gap is second order in σ —doubling σ multiplies it by **4.122849** (Section 3). Fifth, the mean itself does not depend on σ —it stays **110.517091808** (Section 3). Sixth, the separator is whether one is looking at $E[S]$ or $E[\ln S]$ (Section 4). **Keywords:** Itô's lemma, geometric Brownian motion, lognormal distribution, mean and median, Black-Scholes formula.

Scope of this paper (scope note): No new theorem or law is claimed—Itô’s lemma, the lognormal distribution of geometric Brownian motion and the Black-Scholes formula are all standard. No pricing formula is derived—option valuation, risk-neutral measures and replicating portfolios are not the subject; only the distribution of the underlying. Itô’s 2 is not re-derived—its anatomy belongs to Paper 132; the result is borrowed and its financial consequence measured. Estimation is excluded— μ and σ are taken as given, with no estimation error, time-varying volatility or jumps. No investment advice—which quantity one ought to look at is not the subject; only that the two differ, shown in numbers. Relation to earlier papers: Paper 132 showed that the 2 in Itô’s lemma is not a dimension—this paper measures where that 2 appears in the distribution of prices. Paper 292 showed that only the logarithm adds—which is why the median belongs to the logarithmic side. Paper 484 showed that maximising expectation goes bust—the quantity at work there is this same $\sigma^2/2$. Paper 481 showed that without separators a refutation has no destination—this paper writes one of them. What is added is giving mean and median as 110.517091808 and 108.328706767, confirming the ratio against $\exp(\sigma^2 T/2)$ to 12 digits, confirming the logarithmic growth of 0.080000000000 by inversion, giving the factor 4.122849 for a doubled σ , and confirming that the mean ignores σ .

1. Notation and the Question

Setting: Let a price S_t follow geometric Brownian motion $dS = \mu S dt + \sigma S dB$. Applying Itô’s lemma to $\ln S$ gives

$$d(\ln S) = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dB$$

where $-\sigma^2/2$ comes from the second derivative of $\ln$ and from $(dB)^2 = dt$. Below $S_0 = 100$, $\mu = 0.10$, $\sigma = 0.20$ and $T = 1$.

Observation 1.1

The $-\sigma^2/2$ is sometimes read as a correction. Discounting for an aversion to risk—a reading that makes it about preferences or discount rates.

Observation 1.2

But the term contains no preference at all. It comes only from the second derivative of $\ln$ and from $(dB)^2 = dt$ —as Paper 132 showed, the 2 is not a dimension but the consequence of stopping at second order.

Question: Where does that term appear in the distribution of prices.

2. It Appears as the Gap Between Mean and Median

Computation 2.1: Evaluate three quantities from the distribution at $T = 1$.

Quantity	Value
Mean $E[S_T] = S_0 e^{\mu T}$	110.517091808
Median $S_0 e^{(\mu - \sigma^2/2)T}$	108.328706767
Ratio	1.020201340
$\exp(\sigma^2 T/2)$	1.020201340

Observation 2.2

The ratio agrees with $\exp(\sigma^2 T/2)$.1.020201340027 against 1.020201340027—to 12 digits: not close but the same.

Observation 2.3

The growth of the mean logarithm equals $\mu - \sigma^2/2$.0.080000000000 against 0.080000000000—returning on inversion, and equal to $\mu = 0.10$ less $\sigma^2/2 = 0.02$.

Observation 2.4

So Itô’s term is exactly the gap between two summary values.The $-\sigma^2/2$ in the equation—on the distribution, it fixes how far the mean and the median stand apart.

Observation 2.5

The gap is 2.020134 per cent.At $\sigma = 0.20$ and $T = 1$ —small-looking, but a quantity that compounds.

3. The Gap Is Second Order, and the Mean Ignores σ

Computation 3.1: Vary σ .

σ	Gap	Mean
0	0	110.517091808
0.20	2.020134 per cent	110.517091808
0.40	4.122849 times	110.517091808

Observation 3.2

Doubling σ roughly quadruples the gap. A factor of 4.122849—the signature of a second-order dependence, exceeding 4 because of higher terms in the exponential.

Observation 3.3

At $\sigma = 0$ the gap vanishes. The other side of the boundary—without fluctuation, mean and median coincide.

Observation 3.4

The mean itself does not depend on σ . It stays 110.517091808—what moves is the median.

Observation 3.5

So “more volatility lowers the expectation” is wrong. What falls is the median—the mean is constant while the distribution grows a long right tail.

4. The Separator

The separator: is one looking at $E[S]$, or at $E[\ln S]$.

Observation 4.1

Looking at $E[S]$, σ does not enter. 110.517091808—fixed by μ alone.

Observation 4.2

Looking at $E[\ln S]$, $\sigma^2/2$ is subtracted. 0.080000000000—Itô's term appearing directly.

Observation 4.3

The two are distinct roots by the criterion of Paper 300. Both may be called “the expected value”—one an average of prices, one of logarithms: what was taken before averaging differs.

Observation 4.4

It is the quantity at work in Paper 484. Where maximising expectation went bust—there too $\sigma^2/2$ separated the expectation side from the growth side.

Observation 4.5

As Paper 292 said, only the logarithm adds. Prices move by multiplication—so an additive average departs from the multiplicative world's summary value.

Observation 4.6

This is one of the separators of Paper 481. When “the price did not reach the forecast” is reported—whether μ was misjudged or mean and median were confused sends the repair elsewhere.

Observation 4.7

To be honest—the distribution is assumed lognormal. As a consequence of geometric Brownian motion—real prices have jumps and time-varying volatility, and then the closed form $\exp(\sigma^2 T/2)$ does not survive. What survives is only the direction: taking an additive average in a multiplicative world introduces a gap.

5. Roll Call and Closing

1. Mean **110.517091808** (Computation 2.1): fixed by μ alone.
2. Median **108.328706767** (Computation 2.1): $\sigma^2/2$ subtracted.
3. Ratio matches to **12 digits** (Observation 2.2): the core.
4. Logarithmic growth **0.080000000000** (Observation 2.3): returning on inversion.
5. A gap of **2.020134** per cent (Observation 2.5): it compounds.
6. **4.122849** for a doubled σ (Observation 3.2): second order.
7. Vanishing at $\sigma = 0$ (Observation 3.3): the other side.
8. The mean ignores σ (Observation 3.4): the median moves.
9. The separator is $E[S]$ or $E[\ln S]$ (Observation 4.3): distinct roots by Paper 300.
10. Lognormality is assumed (Observation 4.7): jumps break the form.

Closing: The $-\sigma^2/2$ of Black-Scholes is sometimes read as discounting for an aversion to risk—a reading that makes it about preferences. But the term contains no preference at all, coming only from the second derivative of $\ln$ and from $(dB)^2 = dt$; as Paper 132 showed, the 2 is not a dimension but the consequence of stopping at second order. It appears as the gap between mean and median—at $S_0 = 100$, $\mu = 0.10$, $\sigma = 0.20$ and $T = 1$, the mean is **110.517091808**, the median **108.328706767** and their ratio **1.020201340**. That ratio agrees with $\exp(\sigma^2 T/2)$ to 12 digits: not close but the same. The growth of the mean logarithm is **0.080000000000**, exactly $\mu - \sigma^2/2$, and it returns on inversion. So the $-\sigma^2/2$ in the equation fixes, on the distribution, how far the mean and the median stand apart. The gap is 2.020134 per cent: small-looking, but a quantity that compounds. The gap is second order, and the mean ignores σ —doubling σ multiplies the gap by **4.122849**, the signature of a second-order dependence, exceeding 4 because of higher terms in the exponential. At $\sigma = 0$ the gap vanishes: without fluctuation, mean and median coincide. The mean itself stays 110.517091808 throughout, and what moves is the median. So “more volatility lowers the expectation” is wrong—what falls is the median, while the mean is constant and the distribution grows a long right tail. One thing separates them—is one looking at $E[S]$, or at $E[\ln S]$. Looking at $E[S]$, σ does not enter and μ alone fixes it. Looking at $E[\ln S]$, $\sigma^2/2$ is subtracted and Itô’s term appears directly. The two are distinct roots by the criterion of Paper 300: both may be called “the expected value”, but one averages prices and the

other logarithms. It is the quantity at work in Paper 484, where maximising expectation went bust and $\sigma^2/2$ separated the expectation side from the growth side. As Paper 292 said, only the logarithm adds: prices move by multiplication, so an additive average departs from the multiplicative world's summary value. This is one of the separators of Paper 481: when a price does not reach its forecast, whether μ was misjudged or mean and median were confused sends the repair elsewhere. To be honest—the distribution is assumed lognormal, as a consequence of geometric Brownian motion. Real prices have jumps and time-varying volatility, and then the closed form $\exp(\sigma^2 T/2)$ does not survive. What survives is only the direction: taking an additive average in a multiplicative world introduces a gap.

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