

The Yang–Mills Mass Gap and Dimensional Transmutation

— 3rd Edition: An Invariant Built from Conventions, and the Fragility of Its Number — [Paper 05]

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— *From thinking to syncing.* —

ABSTRACT

Classical Yang–Mills theory is **scale invariant**: its action contains no parameter of mass dimension. The quantum theory nevertheless generates a mass scale — this is **dimensional transmutation**, and it is the origin of the mass gap. This paper writes out the standing of the numbers appearing in that mechanism. **First, Λ_{QCD} is an invariant built from two conventions.** The renormalisation scale μ is a **convention**, and the coupling $g(\mu)$ depends on it. **Yet their combination $\Lambda = \mu \exp(-8\pi^2/(b_0 g^2))$ does not depend on μ** (verified numerically here: moving μ from **91** GeV to **1** GeV leaves $\Lambda = 245.75$ MeV unchanged). **Paper 35 separated a product of "fact $\times$ convention"; this paper is the converse case — conventions combining into something convention-independent.** **Second, there are at least three ways a scale gets fixed.** (a) Dimensional analysis (the Planck length of Paper 39, the Kolmogorov length of Paper 03: power laws whose exponents are set by the determinant of the dimension matrix); (b) a choice of units (Papers 35, 38: convention); (c) **dimensional transmutation** (this paper: e^{-1/g^2} , carrying an **essential singularity** and invisible to perturbation theory). **These are distinct roots and must not be merged.** **Third, this number is exponentially fragile.** Since $d \ln \Lambda / d \alpha_s = 2\pi / (b_0 \alpha_s^2) \approx 50$, a 1% error in α_s becomes roughly a 6% error in Λ . This is the opposite of the n^* of Paper 04, which moved by only **1.3** when its input was changed by a factor of **2.5**. **Logarithmically robust numbers and exponentially fragile ones coexist in this series.** **Fourth, the mass gap is a fact and Λ_{QCD} is bookkeeping.** The lightest scalar glueball on the lattice, $m(0^{++}) \approx 1.7$ GeV, is a **measurable physical quantity**, whereas Λ_{QCD} depends on scheme, n_f and loop order (**246** MeV at $n_f = 3$, **153** at $n_f = 4$, **88** at $n_f = 5$). **What is unsolved is not physical existence but mathematical rigour.** **Fifth, the one-loop Λ sits where one loop breaks down.** Λ is defined as the pole of the running coupling, but near it α_s exceeds one (**3.5** at $\mu = 0.3$ GeV). **The definition lies outside its own domain of validity.** **Scope:** a structural account within established quantum field theory (Layer 2). **The existence and mass gap of Yang–Mills theory is a Clay Millennium Prize Problem and is unsolved.** **This paper claims neither its proof, nor the construction of four-dimensional Yang–Mills theory, nor any partial achievement of either.**

Positioning of this paper (discipline): This paper offers a **structural interpretation** of established physical law and mathematical theorems. It claims neither the discovery of new physical laws nor the proof of open problems. The relations shown are transformations of known expressions; the aim is to propose a structural reading of them. No claim contradicting established theory is made. **The appearance of the same number in several contexts is not treated as evidence of a common origin** (Papers 48, 86, 100, 115; §6 applies this explicitly to the number **246**). **The existence and mass gap of Yang–Mills theory is unsolved and nothing here changes that.**

1. Introduction: the classical theory has no mass scale

The classical Yang–Mills action

$$S = -\frac{1}{2g^2} \int d^4x \operatorname{tr} F_{\mu\nu} F^{\mu\nu}$$

is **scale invariant**. In four dimensions the coupling g is dimensionless, and **not a single parameter of mass dimension occurs**.

The quantum theory nevertheless generates a mass scale. Lattice computations give the lightest excitation — the scalar glueball 0^{++} — a finite mass of about **1.7 GeV**. **Where did it come from?**

The answer is dimensional transmutation.

This paper writes out the numbers in that mechanism from the standpoint of which are conventions and which are facts.

2. Dimensional transmutation — a mass from a dimensionless coupling

Theorem 2.1 (Λ_{QCD} and the one-loop geometric measure)

Integrating the one-loop beta function

$$\mu \frac{dg}{d\mu} = -\frac{b_0 g^3}{16\pi^2}, \quad b_0 = \frac{11N - 2n_f}{3}$$

makes the theory generate a renormalisation-group-invariant mass scale

$$\Lambda_{\text{QCD}} = \mu \exp\left(-\frac{8\pi^2}{b_0 g^2(\mu)}\right) = \mu \exp\left(-\frac{2\pi}{b_0 \alpha_s(\mu)}\right)$$

(with $g^2 = 4\pi\alpha_s$). The $16\pi^2$ in the denominator (and the $8\pi^2 = 16\pi^2/2$) is the quantity Paper 46 identified as the one-loop measure of four-dimensional Euclidean momentum space — one of the addresses in the roll call of Paper 09 (3rd edition).

Corollary 2.2 (Numerical evaluation)

With $N = 3$, $\mu = M_Z = 91.19$ GeV and $\alpha_s(M_Z) = 0.118$:

n_f	b_0	Λ_{QCD}
3	9.0	246 MeV
4	8.3	153 MeV
5	7.7	88 MeV

At the glueball scale of 1.7 GeV three light quarks run, so $n_f = 3$ is adopted. The strong dependence on n_f is treated in §5.

3. Λ_{QCD} is an invariant built from two conventions

Established Fact 3.1 (Λ does not depend on the renormalisation scale)

Moving μ makes $g(\mu)$ run, but **their combination Λ does not move**. Using the one-loop running $1/\alpha_s(\mu) = 1/\alpha_s(\mu_0) + (b_0/2\pi) \ln(\mu/\mu_0)$ with $n_f = 3$:

μ [GeV]	$\alpha_s(\mu)$	Λ [MeV]
91.19	0.1180	245.75
10.00	0.1884	245.75
2.00	0.3330	245.75
1.00	0.4974	245.75

Moving μ by a factor of 91 leaves Λ unchanged. This is not an approximation: Λ is by definition the integration constant of the beta function.

Corollary 3.2 (The converse of Paper 35)

Paper 35 (3rd edition) showed that the two 4π 's inside $4\pi G$ are **a fact (Gauss's flux) and a convention (a choice of units)**, and **must be separated**.

The Λ_{QCD} of this paper is the converse case. μ is a **convention** (at which scale one measures the coupling is arbitrary) and $g(\mu)$ depends on it — **taken singly, both are conventions**. Yet their combination **does not depend on either**.

Convention $\times$ convention can make an invariant. "It contains a convention, therefore it has no content" does not follow: **the work is to check what does not depend on the convention**.

4. There are at least three ways a scale gets fixed

This series has now treated three distinct mechanisms by which **a quantity comes to have a definite size**. They are **distinct roots**.

Mechanism	Form	Examples
(a) Dimensional analysis	Power law. The exponents come from the inverse of the dimension matrix, whose denominator is the determinant	Planck length $(\hbar G/c^3)^{1/2}$ (Paper 39, $ \det = 2$ hence a square root); Kolmogorov length $(\nu^3/\varepsilon)^{1/4}$ (Paper 03, $ \det = 4$ hence a fourth root)
(b) Choice of units	Convention. The value may be chosen freely	$G = 4\pi$, $m = 4\pi$, $\hbar = \frac{1}{4\pi}$ (Papers 35, 38, 39)
(c) Dimensional transmutation (this paper)	e^{-1/g^2} , with an essential singularity at $g \rightarrow 0$; every order of the perturbative expansion vanishes	$\Lambda_{\text{QCD}} = \mu e^{-2\pi/(b_0\alpha_s)}$

Observation 4.1 ((c) is invisible to perturbation theory in principle)

Because e^{-1/g^2} has an **essential singularity** at $g = 0$, its Taylor expansion about that point **vanishes at every order**. **Dimensional transmutation therefore never appears, however far the perturbative calculation is carried.**

This is **fundamentally different in mechanism** from (a) and (b). (a) emerges once the dimensional ledger is closed; (b) emerges once one chooses. **(c) emerges only non-perturbatively.**

The three must not be lumped together as "ways a scale is determined".

5. This number is exponentially fragile

Corollary 5.1 (1% in α_s becomes 6% in Λ)

From $\Lambda = \mu e^{-2\pi/(b_0\alpha_s)}$,

$$\frac{d \ln \Lambda}{d \alpha_s} = \frac{2\pi}{b_0 \alpha_s^2} = \frac{2\pi}{9 \times 0.118^2} \approx 50.$$

Moving α_s by 1% therefore moves Λ by 6.1%; by 2%, 12.6%; by 5%, 34.4%. **Errors in the input are amplified exponentially.**

Observation 5.2 (The opposite of Paper 04)

Paper 04 (3rd edition) showed that $n^* = \log_2(E/(k_B T \ln 2))$ is insensitive to its inputs (a factor of 2.5 in the energy moves it by 1.3) and set up the discipline: **when the quantity is exponential, report the logarithm.**

The Λ of this paper lies on the other side. Λ is the value of an exponential, so it does not compress the uncertainty of its inputs but receives it magnified.

That Λ moves from 246 to 88 MeV when n_f goes from 3 to 5 is this fragility made visible. Whenever a value of Λ_{QCD} is quoted, the scheme, n_f and loop order must be quoted with it.

Logarithmically robust numbers (Paper 04) and exponentially fragile ones (this paper) coexist in one series. One must not write digits without knowing which side one is on.

6. The mass gap is a fact; Λ_{QCD} is bookkeeping

Established Fact 6.1 (The glueball on the lattice)

In pure Yang–Mills theory ($SU(3)$, no quarks) the lightest excitation is the scalar glueball 0^{++} , with

$$m(0^{++}) \approx 1.7 \text{ GeV}$$

(Morningstar & Peardon 1999). **That a finite gap exists between the vacuum and the lowest excitation is a numerically established fact.**

Corollary 6.2 (Separating fact from bookkeeping)

$m(0^{++})$ is a physical quantity: it can be measured on the lattice and does not depend on the choice of scheme.

Λ_{QCD} , by contrast, is scheme-dependent: its value changes with $\overline{\text{MS}}$ or otherwise, with n_f , and with the loop order retained. **Λ is a device for bookkeeping the physics, not a physical quantity itself.**

Accordingly the ratio $m(0^{++})/\Lambda \approx 6.9$ is also scheme-dependent and is not a meaningful dimensionless quantity. The discipline of Paper 38 (3rd edition) — that dimensionless ratios are the physics — **applies only when both members of the ratio are physical quantities.**

Corollary 6.3 (On the number 246 — the coincidence with Paper 09 is an accident)

The $\Lambda_{\text{QCD}} \approx 246 \text{ MeV}$ of this paper and the Higgs vacuum expectation value $v \approx 246 \text{ GeV}$ appearing in Paper 09 share a numerical value.

This is a complete accident and implies nothing whatsoever. v is the electroweak scale fixed by the Fermi constant; Λ_{QCD} is the pole of the running strong coupling — **different in origin, in units and in order of magnitude.**

Moreover, as the previous section showed, the value 246 for Λ depends strongly on n_f and on the scheme, ranging from 88 to 246 MeV. The very number that coincides is not itself firm.

This series has repeatedly written, of 4π and of $16\pi^2$, that the same number is not evidence of the same origin (Papers 48, 86, 100, 115). The same discipline is applied here, explicitly, to 246.

What is unsolved is mathematical rigour, not physical existence. The problem as set by Jaffe–Witten (2000) has two stages, and **the first alone — constructing quantum Yang–Mills theory on $\mathbb{R}^4$ so as to satisfy the Wightman axioms — is unsolved.** The content of the problem is: prove, within axiomatic field theory and in the continuum limit, what is numerically visible on the lattice.

7. The one-loop Λ sits where one loop breaks down

Observation 7.1 (The definition lies outside its own domain of validity)

Λ is defined as the point at which the right-hand side of the running relation

$$\frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_s(\mu_0)} + \frac{b_0}{2\pi} \ln \frac{\mu}{\mu_0}$$

vanishes — that is, where α_s diverges. Indeed (with $n_f = 3$):

μ [GeV]	α_s	
2.00	0.333	perturbation theory applies
1.00	0.497	marginal
0.50	0.983	reaches one
0.30	3.50	outside perturbation theory

$\Lambda = 0.246$ GeV lies in a region where α_s has already far exceeded one. That is, the one-loop Λ is defined as the place where the one-loop approximation has ceased to be valid.

This is not an error but the character of Λ . It names, in the language of perturbation theory, the scale at which perturbation theory breaks down; the precise meaning of its value is settled outside perturbation theory. In the vocabulary of Paper 113 it is a module boundary — the place where α_s crosses one.

8. Agreement in form with Paper 45, and the roll call of $16\pi^2$

Rewriting the exponent of Theorem 2.1 with the loop expansion parameter $\alpha_{\text{eff}} \equiv g^2/16\pi^2$,

$$\Lambda_{\text{QCD}} = \mu \exp\left(-\frac{1}{2b_0 \alpha_{\text{eff}}}\right),$$

while the BPST instanton weight treated in Paper 45 is

$$e^{-S_{\text{inst}}} = \exp\left(-\frac{8\pi^2}{g^2}\right) = \exp\left(-\frac{1}{2\alpha_{\text{eff}}}\right).$$

The two have the same functional form, differing only by the factor b_0 . Both carry an essential singularity at $\alpha_{\text{eff}} = 0$ and are therefore never captured by perturbation theory.

This agreement, however, holds within known quantum field theory and was not derived here. The contribution of this paper is limited to **making explicit that the same geometric quantity $16\pi^2$ as in Papers 45, 46 and 47 also appears at the origin of the mass gap.**

This paper's place in the roll call of $16\pi^2$: Paper 09 (3rd edition) listed five addresses of $16\pi^2$ and gave the denominator of this paper's beta function as **"the same measure as there"**. **This paper confirms that** — both are the one-loop measure of four-dimensional momentum space, and both are **a different object from the $16\pi^2$ of Paper 35** (fact $\times$ convention).

Personal reflection of the author (Layer 3; labelled non-scientific / no claim of mechanism)

In the first edition I built a quantity $64\pi^2 = 4 \times 16\pi^2$ and wrote that it governed the mass gap. When the second edition withdrew it, the lesson I recorded was that **"I had reached the right quantity and then lost sight of it by multiplying it by four"**.

Writing the 3rd edition I saw that the lesson had not gone far enough. **I had thought I "found" $16\pi^2$, but it had been written in the denominator of the beta function from the start.** All I had done was multiply it by a factor of my own.

And this time I found something more dangerous. $\Lambda_{\text{QCD}} \approx 246$ MeV, and the $v \approx 246$ GeV of Paper 09. For a moment I saw meaning in the coincidence. **It is exactly the shape of the trap I have fallen into repeatedly with 4π .** And this time it was worse, because **the coinciding 246 is not even a firm number: change n_f and it becomes 88, or 153.**

Wanting to see meaning in a coincidence is a habit of mine. So I write the discipline into the series and then apply the discipline to myself. This time I gave a section of the body to stating plainly that **246 is an accident.** One stops at the edge of the cliff (Paper 101).

9. Scope / Conclusion

What is shown: that dimensional transmutation generates $\Lambda_{\text{QCD}} = \mu e^{-2\pi/(b_0\alpha_s)}$ (§2) / that Λ is independent of μ , an instance of **conventions combining into an invariant** (§3, the converse of Paper 35) / that scales are fixed by **three distinct mechanisms** — dimensional analysis, choice of units, dimensional transmutation — and that the third is **invisible to perturbation theory by virtue of an essential singularity** (§4) / that Λ is **exponentially fragile**, a 1% error in α_s giving 6% in Λ , the opposite of Paper 04 (§5) / that $m(0^{++})$ is physical while Λ is bookkeeping, and that the coincidence at 246 is an accident (§6) / that the one-loop Λ lies outside the one-loop domain of validity (§7) / and the agreement in form with Paper 45 together with this paper's place in the roll call of $16\pi^2$ (§8).

What is not shown (restraint): no proof of the Yang–Mills mass gap, no construction of four-dimensional Yang–Mills theory, no partial achievement of either — all outside the scope of this paper, which contributes nothing to constructive field theory / no link between the 246 of Λ_{QCD} and that of $v / m(0^{++})/\Lambda$ is not treated as a meaningful dimensionless

quantity (since Λ is scheme-dependent) / **the three mechanisms of scale generation are not merged** / no new physical law and no prediction (Layer 2).

The existence and mass gap of Yang–Mills theory is unsolved. Nothing here changes that.

What the 3rd edition adds is the standing of the numbers appearing in dimensional transmutation. Λ_{QCD} is an invariant built from two conventions, arising from a mechanism invisible to perturbation theory in principle, exponentially fragile, and a device for bookkeeping rather than a physical quantity.

And the mass gap itself is a fact. The lattice returns 1.7 GeV.

What is unsolved is proving it within axiomatic field theory — not its existence.

Revision Record

3rd edition (August 2026): removed the word "Withdrawal" from the title, which now names **what the paper shows**; the list of withdrawals is collected in this section.

Added in the 3rd edition: numerical confirmation that Λ is independent of μ (moving μ from 91 GeV to 1 GeV leaves 245.75 MeV), presented as **conventions combining into an invariant**, the converse of Paper 35 (§3); a table of the **three distinct roots by which a scale is fixed** (dimensional analysis, choice of units, dimensional transmutation), noting that the third **vanishes at every perturbative order** by virtue of an essential singularity (§4); the result that $d \ln \Lambda / d\alpha_s \approx 50$, so a 1% error in α_s gives 6% in Λ , and that this is **the opposite of Paper 04's logarithmic robustness** (§5); a separation of $m(0^{++})$ (physical) from Λ (scheme-dependent bookkeeping), with the ratio $m(0^{++})/\Lambda$ declined as a meaningful dimensionless quantity (§6); an explicit statement, applying the discipline of Papers 48, 86, 100 and 115, that **the coincidence between $\Lambda_{\text{QCD}} \approx 246$ MeV and Paper 09's $v \approx 246$ GeV is entirely accidental** (§6); the observation that **the one-loop Λ lies in the region $\alpha_s > 1$, its definition falling outside its own domain of validity** (§7); and confirmation of **this paper's place in the $16\pi^2$ roll call of Paper 09 (3rd edition)** (§8).

Withdrawn in the 2nd edition (July 2026): the first edition's title claimed "a proof of the Yang–Mills mass gap" and its abstract offered "a resolution of the Millennium Prize Problem". **That claim was withdrawn in full.**

- 1. The main theorem was not proved.** Item (iv) of the first edition's proof summary — "the energy to create the lightest one-particle state is bounded below by $\Delta \geq 4\pi > 0$ " — **is the very thing to be proved**, and no argument was given from (i)–(iii) to (iv).

2. **The construction the Millennium problem requires was absent.** Jaffe–Witten (2000) first require **the mathematical construction of quantum Yang–Mills theory on $\mathbb{R}^4$ satisfying the Wightman axioms, and that first stage alone is unsolved.** The first edition discussed the spectrum of a Hamiltonian **H** without saying on which Hilbert space, or how, **H** is defined.
3. **The $64\pi^2$ and the infrared cutoff.** The first edition's $\mathcal{C}_{\text{YM}} = \sum_{\mu=0}^3 (4\pi)^2 = 64\pi^2$ was called an "orthogonal product" while the expression is a **sum**, and no correspondence to any standard quantity of Yang–Mills theory was given. The cutoff $\Lambda_{\text{IR}} = 4\pi/64\pi^2$ likewise had no basis — dividing a mass by a dimensionless "buffer capacity" to obtain a momentum cutoff — and the value obtained is the reciprocal of the "wavelengths $\lambda > 16\pi$ are blocked" stated in the text. **What actually occurs in Yang–Mills theory is $16\pi^2$ (the one-loop measure), not $64\pi^2$.**

First edition: Zenodo 10.5281/zenodo.21404236.

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