

Three Separate Answers Are All Decided Where Something Is Tangent

— The Legendre transform’s contact point solves $f'(x) = p$, giving $x = \mathbf{2.500000}$ and a conjugate of $\mathbf{6.250000}$ at $p = 5$ — the common tangent touches at $v = \pm 1$ with slope $\mathbf{0.300000000}$, and between them the curve falls below it at $\mathbf{0}$ points — [Paper 507]

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September 2026

— *From thinking to syncing.* —

Abstract

A transform, a bound and a phase coexistence are taken to be separate tools. This paper shows that all three have their answers decided where something is tangent—the location of the contact point is the answer. No new theorem or law is claimed. First, the Legendre transform’s contact point solves $f'(x) = p$, giving $x = \mathbf{2.500000}$ and a conjugate of $\mathbf{6.250000}$ at $p = 5$ (Section 2). Second, for a convex function two transforms return the original— $\mathbf{9.000000000}$ against $\mathbf{9.000000000}$ (Section 2). Third, the Chernoff bound’s best exponent is $\mathbf{0.082282879}$, matching the Kullback-Leibler divergence (Section 3). Fourth, its contact point is $\lambda^* = \ln(a/(1-a)) = \mathbf{0.847298}$ —move off it and the exponent falls (Section 3). Fifth, this is the core of the paper. The common tangent touches at $v = \pm 1$ with slope $\mathbf{0.300000000}$, and between them the curve falls below it at $\mathbf{0}$ points (Section 4). Sixth, the separator is whether the tangency is at one point or two (Section 5). **Keywords:** Legendre transform, Chernoff bound, common tangent construction, convex analysis, duality.

Scope of this paper (scope note): No new theorem or law is claimed—the Legendre transform, the Chernoff bound and the common-tangent construction are all standard. No convex analysis is written—subdifferentials, duality and general convex sets are not entered; only the smooth one-dimensional case. The equal-area rule is not computed—Maxwell’s equal-area rule is a statement on the pressure-volume curve, while this paper treats only the common tangent on the free energy. No thermodynamics is built—coexistence, latent heat and real equations of state are not the subject; one double-well function is used. A fourth was not counted—Gibbs’s variational principle is said to have the same shape, but no numbers are given for it here. Three were counted. Relation to earlier papers: Paper 487 showed that a stationary point comes in three kinds—that classifies what one-point tangency touches; this counts how many points it touches. Paper 486 showed that optima in three fields are one vertex—likewise a case of separate-looking answers arising from one operation. Paper 300 showed that a shared root can be decided—one-point and two-point tangency are distinct roots. Paper 481 showed that without separators a refutation has no destination—this paper writes one of them. What is added is placing the three contact points in one table, confirming the conjugate and the double transform numerically, confirming that the Chernoff contact point is $\ln(a/(1-a))$, matching the common tangent to its exact solution at 10^{-6} , and writing the convexification test as a count of points falling below.

1. Notation and the Question

Setting: Examine where the answer comes from in three settings. The Legendre transform is

$$f^*(p) = \max_x [px - f(x)]$$

whose maximiser satisfies $f'(x) = p$. The Chernoff bound is $\max_\lambda [\lambda a - \ln M(\lambda)]$, and a common tangent is a line touching at two points with the same slope. Below, $f(x) = x^2$, a fair coin with $a = 0.7$, and $g(v) = (v^2 - 1)^2 + 0.3v$.

Observation 1.1

A transform, a bound and a phase coexistence are taken to be separate tools. A change of variables in thermodynamics, an inequality in probability, a phase diagram of matter—different fields, different purposes.

Observation 1.2

But all three take a maximum or a minimum. $\max_x$, $\max_\lambda$, a shared slope—and the condition for an extremum is tangency.

Question: Where does the contact point fall, and how does it become the answer.

2. The Legendre Transform: Tangent at One Point

Computation 2.1: Compute the conjugate of $f(x) = x^2$ numerically.

p	$f^*(p)$	Exact $p^2/4$	Contact point x
2.0	1.000000	1.000000	1.000000
5.0	6.250000	6.250000	2.500000

Observation 2.2

The conjugate matches the closed form.6.250000000 against 6.250000000—an independent check.

Observation 2.3

The contact point satisfies $f'(x) = p.f'(2.500000) = 5.000000$ against $p = 5.0$ —at the maximiser, the slope equals p .

Observation 2.4

Two transforms return the original.9.000000000 against 9.000000000—because the function is convex: the family of tangents reconstructs it.

Observation 2.5

So the transform recounts the tangents.From a set of points to a set of tangent lines—and the answer is fixed by where they touch.

3. The Chernoff Bound: Choosing the Tangent Exponent

Computation 3.1: For a fair coin at $a = 0.7$, scan λ for the best exponent.

Quantity	Value
Best λ (numerical)	0.847300
Contact point $\ln(a/(1-a))$ (closed form)	0.847298
Best exponent	0.082282879
Kullback-Leibler divergence	0.082282879

Observation 3.2

The best exponent matches the divergence.0.082282879 against 0.082282879—an optimisation whose answer rewrites itself in the language of information.

Observation 3.3

The contact point matches its closed form.0.847300 against 0.847298—the difference is the scan’s step of 10^{-5} .

Observation 3.4

At $\lambda = 0$ the bound becomes trivial.0.000000000000—the other side of the boundary: an inequality saying nothing.

Observation 3.5

Away from the best λ the exponent falls.0.062460777 at half and 0.016145752 at double—both below 0.082282879, so it really is the maximum.

Observation 3.6

The Legendre transform is at work here too. $\max_{\lambda}[\lambda a - \ln M(\lambda)]$ —the conjugate of the cumulant generating function, whose contact point is the best λ .

4. The Common Tangent: Tangent at Two Points

Computation 4.1: Find the common tangent of $g(v) = (v^2 - 1)^2 + 0.3v$ numerically and compare with the exact solution.

Quantity	Numerical	Exact
Left contact point	-1.000000	-1
Right contact point	1.000000	+1
Slope	0.300000000	0.300000000
Residual	2.393×10^{-10}	—

Observation 4.2

The contact points match the exact solution. ± 1 —adding the line $0.3v$ does not move them; only the slope becomes 0.3.

Observation 4.3

The slopes agree at two points. A residual of 2.393×10^{-10} —not one point of tangency but two, separated by 2.000000.

Observation 4.4

Between them the curve never falls below the tangent. 0 points—a convexification: the interval where two phases coexist.

Observation 4.5

Change the slope and it does fall below. At 1.5 times, 3358 points, the deepest at -0.300000 —the test responds to the slope, so being a common tangent is doing the work.

Observation 4.6

At both ends the tangent meets the curve. 0.300000000 against 0.300000000 —expected from tangency, but confirmed independently.

5. The Separator

The separator: is the tangency at one point, or at two.

Observation 5.1

At one, and the answer is a number. $f^*(p)$, the Chernoff exponent—move the contact point and the answer moves.

Observation 5.2

At two, and the answer is an interval. $v \in [-1, 1]$ —throughout which the same slope is shared.

Observation 5.3

The two are distinct roots by the criterion of Paper 300. Both are “decided where something is tangent”—but one returns a point and the other an interval: they differ in the type of what they return.

Observation 5.4

Paper 487 divided the one-point side into three. Minimum, maximum, degenerate—that classifies what the tangency touches, while this counts how many points it touches: a two-level classification.

Observation 5.5

It has the shape of Paper 486 too. Optima in three fields at one vertex—separate-looking answers arising from one operation.

Observation 5.6

This is one of the separators of Paper 481. When “the optimisation gives the wrong answer” is reported—whether the contact point was misidentified or one- and two-point tangency were confused sends the repair elsewhere.

Observation 5.7

To be honest—three were counted, and the equal-area rule was not computed. Maxwell’s rule is a statement on the pressure-volume curve—its correspondence with the common tangent on the free energy is known, but this paper does not verify it numerically. Gibbs’s variational principle is said to have the same shape, and no numbers are given for it either.

6. Roll Call and Closing

1. $f^*(5) = 6.250000$ (Computation 2.1): **contact point $x = 2.500000$.**
2. The contact point solves $f'(x) = p$ (Observation 2.3): **the slope equals p .**
3. Two transforms return the original (Observation 2.4): **9.000000000.**
4. Chernoff exponent **0.082282879** (Computation 3.1): **matching the divergence.**
5. Contact point $\lambda^* = 0.847298$ (Observation 3.3): **$\ln(a/(1-a))$.**
6. Moving off it lowers the exponent (Observation 3.5): **really the maximum.**
7. Common tangent slope **0.300000000** (Computation 4.1): **the core.**
8. **0 points fall below** (Observation 4.4): **a convexification.**
9. The separator is one point or two (Observation 5.3): **distinct roots by Paper 300.**
10. The equal-area rule was not computed (Observation 5.7): **three were counted.**

Closing: A transform, a bound and a phase coexistence are taken to be separate tools—a change of variables in thermodynamics, an inequality in probability, a phase diagram of matter. But all three take a maximum or a minimum, and the condition for an extremum is tangency. The Legendre transform: tangent at one point—the conjugate of $f(x) = x^2$ is 1.000000 at $p = 2.0$ and **6.250000** at $p = 5.0$, matching the closed form $p^2/4$. The contact point at $p = 5.0$ is $x = 2.500000$, where $f'(2.500000) = 5.000000$: at the maximiser the slope equals p . Two transforms return 9.000000000 against 9.000000000, because the function is convex and the family of tangents reconstructs it. So the transform recounts the tangents, moving from a set of points to a set of tangent lines, and the

answer is fixed by where they touch. The Chernoff bound: choosing the tangent exponent—for a fair coin at $a = 0.7$, the best λ is **0.847300** and the best exponent **0.082282879**, matching the Kullback-Leibler divergence. The contact point matches $\ln(a/(1 - a)) = \mathbf{0.847298}$, differing only by the scan's step of 10^{-5} . At $\lambda = 0$ the bound is 0.000000000000 and says nothing, the other side of the boundary. Away from the best λ the exponent falls to 0.062460777 at half and 0.016145752 at double. The Legendre transform is at work here too: the conjugate of the cumulant generating function, whose contact point is the best λ . The common tangent: tangent at two points—for $g(v) = (v^2 - 1)^2 + 0.3v$, numerical and exact agree on contact points $v = \pm 1$, slope **0.300000000**, residual 2.393×10^{-10} . Adding the line $0.3v$ does not move the contact points; only the slope becomes 0.3. The slopes agree at two points separated by 2.000000, so this is not one point of tangency but two. Between them the curve never falls below the tangent, at 0 points: a convexification, the interval where two phases coexist. Change the slope to 1.5 times and 3358 points fall below, the deepest at -0.300000 , so the test responds to the slope. One thing separates them—is the tangency at one point, or at two. At one, the answer is a number, and moving the contact point moves the answer. At two, the answer is an interval, throughout which the same slope is shared. The two are distinct roots by the criterion of Paper 300: both are “decided where something is tangent”, but one returns a point and the other an interval. Paper 487 divided the one-point side into minimum, maximum and degenerate—that classifies what the tangency touches, while this counts how many points it touches, a two-level classification. It has the shape of Paper 486 too, where separate-looking answers arose from one operation. This is one of the separators of Paper 481: when an optimisation gives the wrong answer, whether the contact point was misidentified or one- and two-point tangency were confused sends the repair elsewhere. To be honest—three were counted, and the equal-area rule was not computed. Maxwell's rule is a statement on the pressure-volume curve, and its correspondence with the common tangent on the free energy is known but not verified numerically here. Gibbs's variational principle is said to have the same shape, and no numbers are given for it either.

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