

The Threshold and the Final Size Both Come from R_0 , and Their Ratio Is Not Constant

— At $R_0 = 2$ the threshold $1 - 1/R_0$ is **0.500000** and the final size **0.796812** — the threshold comes from a linearisation and the final size from the nonlinear equation $z = 1 - e^{-R_0 z}$, so the two are not one quantity —

[Paper 511]

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— *From thinking to syncing.* —

Abstract

The threshold and the final size, both fixed by R_0 , are readily taken to say one thing. This paper shows that their ratio changes with R_0 —running from **1.7484** to **1.2413**. No new theorem or law is claimed. First, at $R_0 = 2$ the threshold is **0.500000** and the final size **0.796812** (Section 2). Second, inverting the final size returns R_0 —**2.000000000** against **2.0** (Section 2). Third, at $R_0 = 1$ and at $R_0 = 0.9$ the final size is **0**—both sides of the boundary were checked (Section 2). Fourth, this is the core of the paper. Their ratio is **1.7484** at $R_0 = 1.5$ and **1.2413** at $R_0 = 5$ (Section 3). Fifth, the larger R_0 , the nearer the ratio to one—both approach one, at different rates (Section 3). Sixth, the separator is whether the quantity came from a linearisation or from a nonlinear equation (Section 4).

Keywords: basic reproduction number R_0 , threshold theorem, final size equation, SIR model, mathematical epidemiology.

Scope of this paper (scope note): No new theorem or law is claimed—the threshold theorem and the final-size equation are both standard. No epidemiological advice—vaccination policy, estimating a real R_0 and the effect of interventions are not the subject; only the relation between two quantities. One model only—a closed, homogeneously mixing SIR model; age structure, space and time-varying contact rates are excluded. No adjudication of priority—to whom, and when, the threshold idea belongs is not treated. Efficacy is excluded—the vaccine efficacy of Paper 401 does not enter; only the immunised fraction itself. Relation to earlier papers: Paper 401 showed that not everyone needs immunity and yet everyone may not be enough—the threshold used there and the final size here are different quantities. Paper 300 showed that a shared root can be decided—two quantities from one R_0 can still be distinct roots. Paper 486 showed that optima in three fields are one vertex—there, separate-looking and one-rooted; here, same-looking and distinct-rooted. Paper 481 showed that without separators a refutation has no destination—this paper writes one of them. What is added is tabulating the two by R_0 , solving the final size as 0.796812 and confirming it by inversion, checking both sides of the boundary at $R_0 = 1$ and 0.9, giving the ratio's movement from 1.7484 to 1.2413, and placing the separator at linear versus nonlinear.

1. Notation and the Question

Setting: In a closed, homogeneously mixing population, write R_0 for the basic reproduction number. From the condition that no epidemic starts, the fraction that must be immune is

$$p_c = 1 - \frac{1}{R_0}$$

while the fraction z that has been infected once the epidemic ends is the positive solution of

$$z = 1 - e^{-R_0 z}$$

The first comes from a linearisation valid when infections are few; the second from a balance across the whole epidemic. Below, R_0 is varied and the two set side by side.

Observation 1.1

The threshold and the final size are both fixed by R_0 . Both from one number—so they say one thing, runs the picture.

Observation 1.2

But the expressions differ in form. $1 - 1/R_0$ against $z = 1 - e^{-R_0 z}$ —one closed, the other containing itself.

Question: How far apart are they, and is that distance constant.

2. Setting the Two Side by Side

Computation 2.1: Evaluate both by R_0 .

R_0	Threshold $1 - 1/R_0$	Final size z
0.9	0.000000	0.000000
1.0	0.000000	0.000000
1.5	0.333333	0.582812
2.0	0.500000	0.796812
3.0	0.666667	0.940480

Observation 2.2

The final size at $R_0 = 2$ matches the known value.0.796812—agreeing with the frequently quoted eighty per cent.

Observation 2.3

Inverting returns $R_0. - \ln(1 - z)/z = 2.000000000$ against 2.0—an independent check.

Observation 2.4

At $R_0 = 1$ the final size is zero.The boundary—where the threshold is zero too, so the two coincide.

Observation 2.5

At $R_0 = 0.9$ it is zero as well.The other side—the threshold formula returns a negative value, cut at 0, and the final-size equation has no positive solution at all.

Observation 2.6

The final size exceeds the threshold. $0.796812 > 0.500000$ —because the epidemic stops only after the threshold is crossed.

3. The Ratio Is Not Constant

Computation 3.1: Divide the final size by the threshold.

R_0	Final size / threshold
1.5	1.7484
2.0	1.5936
3.0	1.4107
5.0	1.2413

Observation 3.2

The ratio changes with R_0 . From 1.7484 to 1.2413—were the two expressions of one quantity, it would be constant.

Observation 3.3

The larger R_0 , the nearer to one. 1.2413—both approach one, the threshold more slowly.

Observation 3.4

So “know the threshold and you know the final size” cannot be written as a proportion. R_0 must be passed through—no relation independent of R_0 joins them.

Observation 3.5

The difference is called the overshoot. $0.796812 - 0.500000 = 0.296812$ at $R_0 = 2$ —the epidemic does not halt on reaching the threshold, and this is how far it runs past.

4. The Separator

The separator: did the quantity come from a linearisation, or from a nonlinear equation.

Observation 4.1

From a linearisation, and it has a closed form. $1 - 1/R_0$ —looking only at the moment when infections are few, and containing nothing of what follows.

Observation 4.2

From a nonlinear equation, and it contains itself. $z = 1 - e^{-R_0 z}$ —a balance across the whole epidemic, solvable only numerically.

Observation 4.3

The two are distinct roots by the criterion of Paper 300. Both come from R_0 —but one is a condition for an epidemic to begin and the other a result after it has: they ask about different moments.

Observation 4.4

The moving ratio is the evidence. From 1.7484 to 1.2413—a shared root would admit a conversion independent of R_0 .

Observation 4.5

Paper 401 used only the threshold side. Not everyone needs immunity—what did the work there was $1 - 1/R_0$, not the final size.

Observation 4.6

Paper 486 runs the other way. There, separate-looking and one-rooted—here, same-looking and distinct-rooted. The same test serves in both directions.

Observation 4.7

To be honest—the model is a closed, homogeneously mixing population. No age structure, no space, no varying contact rate—real epidemics usually end below this, so 0.796812 is close to an upper bound. What is shown is a relation between two quantities, not a prediction.

5. Roll Call and Closing

1. Threshold 0.500000 at $R_0 = 2$ (Computation 2.1): **final size 0.796812.**
2. Matching the familiar eighty per cent (Observation 2.2): **an independent check.**
3. Inversion gives 2.000000000 (Observation 2.3): **returning R_0 .**
4. Zero at $R_0 = 1$ and at 0.9 (Observation 2.5): **both sides of the boundary.**
5. Ratio from 1.7484 to 1.2413 (Computation 3.1): **the core.**
6. Nearer to one at larger R_0 (Observation 3.3): **the threshold lags.**
7. An overshoot of 0.296812 (Observation 3.5): **how far it runs past.**
8. The separator is linear or nonlinear (Observation 4.3): **distinct roots by Paper 300.**
9. The reverse of Paper 486 (Observation 4.6): **the test serves both ways.**
10. A homogeneously mixing model (Observation 4.7): **not a prediction.**

Closing: The threshold and the final size are both fixed by R_0 —both from one number, so they say one thing. But the expressions differ in form: $1 - 1/R_0$ against $z = 1 - e^{-R_0 z}$, one closed and the other containing itself. Setting the two side by side—at $R_0 = 2$ the threshold is **0.500000** and the final size **0.796812**, agreeing with the frequently quoted eighty per cent. Computing $-\ln(1 - z)/z$ from the final size returns 2.000000000. At $R_0 = 1$ both are zero and

coincide. At $R_0 = 0.9$ both are zero as well: the threshold formula returns a negative value, cut at 0, and the final-size equation has no positive solution. The final size exceeds the threshold, because the epidemic stops only after the threshold is crossed. The ratio is not constant—dividing gives **1.7484** at $R_0 = 1.5$, **1.5936** at 2.0, **1.4107** at 3.0 and **1.2413** at 5.0. Were the two expressions of one quantity, it would be constant. The larger R_0 , the nearer to one, with the threshold approaching more slowly. So “know the threshold and you know the final size” cannot be written as a proportion: R_0 must be passed through, and no relation independent of it joins them. The difference, called the overshoot, is 0.296812 at $R_0 = 2$ —how far the epidemic runs past its own threshold. One thing separates them—did the quantity come from a linearisation, or from a nonlinear equation. From a linearisation, and it has a closed form: $1 - 1/R_0$ looks only at the moment when infections are few. From a nonlinear equation, and it contains itself: $z = 1 - e^{-R_0 z}$ is a balance across the whole epidemic, solvable only numerically. The two are distinct roots by the criterion of Paper 300: both come from R_0 , but one is a condition for an epidemic to begin and the other a result after it has. The moving ratio is the evidence, since a shared root would admit a conversion independent of R_0 . Paper 401 used only the threshold side, where what did the work was $1 - 1/R_0$. Paper 486 runs the other way—separate-looking and one-rooted there, same-looking and distinct-rooted here—so the same test serves in both directions. To be honest—the model is a closed, homogeneously mixing population, with no age structure, no space and no varying contact rate. Real epidemics usually end below this, so 0.796812 is close to an upper bound. What is shown is a relation between two quantities, not a prediction.

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