

Split the Surplus Three Ways and a Curved Problem Becomes a Flat One

— subtract a third of it from each angle and solve as a plane triangle, and the residual grows as the square of the excess, staying at 4.106×10^{-4} of it even at a million km^2 — [Paper 512]

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September 2026

— *From thinking to syncing.* —

Abstract

A triangle on a curved surface is taken to require curved tools. This paper shows that splitting the surplus three ways lets flat tools suffice—the residual grows as the square and vanishes within practical range. No new theorem or law is claimed. First, the spherical excess is proportional to the area—**0.508171** arcseconds at 100 km^2 , and 10^4 times that for 10^4 times the area (Section 2). Second, at a million km^2 the excess is **1.411586** degrees (Section 2). Third, inverting returns the area—1000000.000000 against 1000000 (Section 2). Fourth, this is the core of the paper. After the split, a hundredfold area multiplies the residual by **10000.000000**—it grows as the square of the excess (Section 3). Fifth, even at a million km^2 the residual is 4.106×10^{-4} of the excess, and at 100 km^2 it is 4.106×10^{-8} (Section 3). Sixth, the separator is whether the surplus is spread out or the surface kept curved (Section 4).

Keywords: spherical excess, Legendre's theorem, spherical triangle, geodesy, triangulation.

Scope of this paper (scope note): No new theorem or law is claimed—the formula for spherical excess and Legendre’s theorem are both standard. No geodesy is built—the flattening of the Earth, survey errors and the practice of triangulation are not the subject; only triangles on a sphere. The second-order coefficient is not derived—a known value is used for the residual’s coefficient; it is not derived here. The angles themselves are not solved—no sides or angles are computed; only the size of the residual. Only a sphere—on an ellipsoid the excess formula changes, and that is not entered. Relation to earlier papers: Paper 409 showed that no undistorted map exists, giving 1.4113 degrees of surplus at a million km²—this paper examines how that surplus can be disposed of. Paper 224 showed that a surface flat wherever it is touched still totals 4π —the same shape: the surplus is small locally and does not vanish globally. Paper 300 showed that a shared root can be decided—solving by spreading and solving on the curved surface are distinct roots. Paper 481 showed that without separators a refutation has no destination—this paper writes one of them. What is added is giving the excess at three areas, confirming by inversion that the area returns, showing the residual growing as the square at 10000.000000, giving the ratios 4.106×10^{-4} and 4.106×10^{-8} , and placing the separator in the manner of approximating.

1. Notation and the Question

Setting: Take a triangle of area A on a sphere of radius R . Its angles sum to more than π , by

$$\varepsilon = \frac{A}{R^2}$$

Legendre’s theorem says that subtracting $\varepsilon/3$ from each angle allows the triangle to be solved as a plane triangle with the same side lengths. Below, $R = 6371$ km, and both the excess and the residual after this treatment are measured.

Observation 1.1

A triangle on a curved surface is taken to require curved tools. Spherical trigonometry—the plane formulas being unusable, runs the picture.

Observation 1.2

But the excess is merely proportional to the area, not scattered among the angles. A single number ε —spread it out and the problem may move to the plane.

Question: How much remains after spreading.

2. The Excess Is Proportional to the Area

Computation 2.1: Evaluate the excess at three areas.

Area	Spherical excess (rad)	Arcseconds
100 km ²	$2.463682790 \times 10^{-6}$	0.508171
10 ⁴ km ²	$2.463682790 \times 10^{-4}$	50.817105
10 ⁶ km ²	$2.463682790 \times 10^{-2}$	5081.710534

Observation 2.2

At a million km² it is 1.411586 degrees.1.4 degrees—the angles of the triangle summing to 181.4 degrees.

Observation 2.3

It is proportional to the area.10000.000000 times for 10⁴ times the area—confirming that the closed form is linear.

Observation 2.4

Inverting returns the area.1000000.000000 against 1000000—an independent check.

Observation 2.5

At 100 km² it is only 0.508171 arcseconds.A triangle some 15 km on a side—measurable even by nineteenth-century surveying, and not small enough to ignore.

3. Split Three Ways, Only the Square Remains

Computation 3.1: Evaluate the residual after subtracting $\varepsilon/3$ from each angle and solving in the plane.

Area	Excess ϵ	Second-order residual	Ratio
100 km ²	2.463683×10^{-6}	1.011622×10^{-13}	4.106×10^{-8}
10 ⁴ km ²	2.463683×10^{-4}	1.011622×10^{-9}	4.106×10^{-6}
10 ⁶ km ²	2.463683×10^{-2}	1.011622×10^{-5}	4.106×10^{-4}

Observation 3.2

The residual grows as the square of the excess. A hundredfold area multiplies it by 10000.000000—while the excess only rises a hundredfold, so the ratio worsens a hundredfold.

Observation 3.3

Even so, the ratio at a million km² is 4.106×10^{-4} . Under one part in 2400 of the excess—spread it once and the remainder vanishes within practical range.

Observation 3.4

At 100 km² the ratio is 4.106×10^{-8} . Four orders smaller—the other side of the boundary: the smaller the triangle, the better.

Observation 3.5

So the quality of the approximation is set by the area. Not by shape or position—a triangle anywhere leaves the same residual for the same area.

Observation 3.6

The split works because it cancels the first-order term exactly. Distributed equally among three angles—subtract it all from one angle instead and a first-order term survives.

4. The Separator

The separator: is the surplus spread out and the problem moved to the plane, or is the surface kept curved.

Observation 4.1

Spread it, and flat tools suffice. The plane sine and cosine rules—at the cost of a second-order error.

Observation 4.2

Keep it curved, and there is no error. Spherical trigonometry—at the cost of different tools.

Observation 4.3

The two are distinct roots by the criterion of Paper 300. Both are “solving a triangle”—one an approximation carrying error and one exact: the presence of error decides it.

Observation 4.4

The surplus Paper 409 reported is this excess. 1.4113 degrees at a million km^2 —the same quantity as this paper’s 1.411586 : there it was shown not to vanish, here it is shown to be spreadable.

Observation 4.5

It has the shape of Paper 224 as well. Flat wherever touched, and four pi in total—the surplus is small locally and does not vanish globally, and the three-way split is the tool that uses that smallness.

Observation 4.6

This is one of the separators of Paper 481. When “the plane calculation did not agree” is reported—whether the spreading was forgotten or the triangle was too large sends the repair elsewhere.

Observation 4.7

To be honest—the second-order coefficient was taken as known, $\varepsilon^2/60$ —it is not derived here. What is shown is that the residual grows as the square and that its size is set by the area, not a verification of the coefficient.

5. Roll Call and Closing

1. **0.508171 arcseconds at 100 km^2** (Computation 2.1): **a measurable size.**
2. **1.411586 degrees at a million** (Observation 2.2): **angles summing to 181.4.**
3. **Proportional to the area** (Observation 2.3): **10000.000000 times.**
4. **Inversion returns the area** (Observation 2.4): **an independent check.**
5. **The residual grows as the square** (Observation 3.2): **the core.**
6. **Ratio 4.106×10^{-4} at a million km^2** (Computation 3.1): **under one part in 2400.**
7. **4.106×10^{-8} at 100 km^2** (Observation 3.4): **four orders smaller.**
8. **The split cancels the first order** (Observation 3.6): **spreading is what works.**
9. **The separator is spread or curved** (Observation 4.3): **distinct roots by Paper 300.**
10. **The coefficient was borrowed** (Observation 4.7): **not derived here.**

Closing: A triangle on a curved surface is taken to require curved tools—spherical trigonometry, the plane formulas being unusable. But the excess is merely proportional to the area, not scattered among the angles: it is a single number ε , and spreading it may move the problem to the plane. The excess is proportional to the area—at $R = 6371$ km it is **0.508171 arcseconds at 100 km^2** , **50.817105 at 10^4** and **5081.710534 at 10^6** . At a million km^2 that is **1.411586 degrees**, so the angles of the triangle sum to 181.4 degrees. It is proportional to the area, rising 10000.000000-fold for 10^4 times the area. Inverting returns 1000000.000000. Even at 100 km^2 it is 0.508171 arcseconds: a triangle some 15 km on a side is measurable by nineteenth-century surveying and not small enough to ignore. Split three ways, only the square remains—subtracting $\varepsilon/3$ from each angle leaves a residual of **1.011622×10^{-13} at 100 km^2** and **1.011622×10^{-5} at 10^6** . A hundredfold area multiplies the residual by **10000.000000** while the excess rises only a hundredfold, so the ratio worsens a hundredfold. Even so, at a million km^2 the ratio is **4.106×10^{-4}** , under one part in 2400 of the excess: spread it once and the remainder vanishes within practical range. At 100 km^2 the ratio is **4.106×10^{-8}** , four orders smaller. So the quality of the approximation is set by the area, not by shape or position: a triangle anywhere leaves the same residual for the same area. The split works because it cancels the first-order term exactly; subtract it all from one angle and a first-order term survives. One thing separates them—is the surplus spread out and the problem moved to the plane, or is the surface kept curved. Spread it, and flat tools suffice—the plane sine and cosine rules—at the cost of a second-order

error. Keep it curved, and there is no error, at the cost of different tools. The two are distinct roots by the criterion of Paper 300: both are “solving a triangle”, but one is an approximation carrying error and one exact. The surplus Paper 409 reported is this excess, 1.4113 degrees at a million km², the same quantity as this paper’s 1.411586; there it was shown not to vanish, here it is shown to be spreadable. It has the shape of Paper 224 as well: flat wherever touched and four pi in total, the surplus small locally and not vanishing globally, with the three-way split the tool that uses that smallness. This is one of the separators of Paper 481: when a plane calculation does not agree, whether the spreading was forgotten or the triangle was too large sends the repair elsewhere. To be honest—the second-order coefficient $\varepsilon^2/60$ was taken as known and is not derived here. What is shown is that the residual grows as the square and that its size is set by the area, not a verification of the coefficient itself.

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