

Not Freed from Units, but Having Handed the Choice to the Covariance

— Multiply the first axis by **100** and the plain distance moves from **5.000000** to **300.026665**, while the covariance-based distance stays at **4.123105626** without moving a digit — set it to the identity and the plain distance **5.000000000000** returns — [Paper 516]

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— *From thinking to syncing.* —

Abstract

A distance measured through a covariance is said to be free of the choice of units. This paper shows that it is not freed; the choice has moved into the covariance—replace the covariance and the value changes. No new theorem or law is claimed. First, the plain distance is **5.000000000000** and the covariance-based one **4.123105626** (Section 2). Second, this is the core of the paper. Multiplying the first axis by 100 moves the plain distance to **300.026665** while the covariance-based one stays at **4.123105625618** (Section 2). Third, yet replacing the covariance gives **3.282952601** (Section 3). Fourth, a correlated covariance gives **2.157439560** (Section 3). Fifth, the identity returns the plain distance, **5.000000000000**—the boundary (Section 3). Sixth, the separator is whether the choice disappeared or moved (Section 4). **Keywords:** Mahalanobis distance, covariance, unit invariance, statistical distance, multivariate analysis.

Scope of this paper (scope note): No new theorem or law is claimed—the Mahalanobis distance and its invariance under changes of unit are both standard. No statistics is built—estimation, outlier detection and discriminant analysis are not the subject; only the value of the distance. High dimensions are excluded—only a two-dimensional example; the difficulty of estimating a covariance in high dimension is not entered. No covariance is preferred—whether to use a sample covariance or prior knowledge is not the subject; only that replacing it changes the value. Positive definiteness is assumed—degenerate covariances are not treated. Relation to earlier papers: Paper 95 separated convention from fact—the covariance here is a quantity that looks like fact and sits on the convention side. Paper 300 showed that a shared root can be decided—freedom from units and dependence on a covariance are distinct roots. Paper 505 showed that the sphere of influence is three spheres with different exponents—both are cases where one apparent quantity turns out to be as many as there are choices. Paper 481 showed that without separators a refutation has no destination—this paper writes one of them. What is added is setting the plain and covariance-based distances side by side, showing that only one moves when a unit is multiplied by 100, giving the values under three replacement covariances, confirming that the identity returns the plain distance, and placing the separator at whether the choice vanished or moved.

1. Notation and the Question

Setting: Measure the distance between two points x and y two ways. The plain distance is

$$d_E = \sqrt{(x - y)^\top (x - y)}$$

and the distance through a covariance Σ is

$$d_M = \sqrt{(x - y)^\top \Sigma^{-1} (x - y)}$$

Below, $x = (0, 0)$ and $y = (3, 4)$, with Σ replaced and axis units changed.

Observation 1.1

A distance measured through a covariance is said to be free of the choice of units. The same value in metres or kilometres—the arbitrariness gone, runs the picture.

Observation 1.2

But Σ appears in the expression. Which covariance to use—that too is a choice.

Question: Has the choice vanished, or only moved.

2. Changing the Units Does Not Move It

Computation 2.1: Evaluate both distances.

Measure	Value
Plain distance	5.000000000000
Through diag(9, 1)	4.123105626

Observation 2.2

The plain distance is 5.A three-four right triangle—a value confirmable independently.

Observation 2.3

Through the covariance it differs.4.123105626—the first axis varies more, so the separation along it is discounted.

Computation 2.4: Multiply the first axis by 100.

Measure	Original units	Units ×100
Plain distance	5.000000	300.026665
Through the covariance	4.123105625618	4.123105625618

Observation 2.5

The plain distance moves sixtyfold.From 5.000000 to 300.026665—on a change of units alone, with the relation between the points untouched.

Observation 2.6

The covariance-based distance does not move a digit.4.123105625618 against 4.123105625618—the covariance scales by the same factor and cancels against its inverse.

Observation 2.7

So far “free of units” causes no trouble.It really does not move—on this operation alone, the arbitrariness looks gone.

3. Replacing the Covariance Does Move It

Computation 3.1: Measure the same two points through other covariances.

Covariance	Distance
diag(9, 1)	4.123105626
diag(1, 9)	3.282952601
Correlated (diagonal 4, off-diagonal 1.5)	2.157439560
Identity	5.000000000000

Observation 3.2

Swapping the axes alone changes the value.From 4.123105626 to 3.282952601—the points unchanged, only the covariance replaced.

Observation 3.3

Correlation changes it further.2.157439560—the off-diagonal term tying the two axes’ separations together.

Observation 3.4

The identity returns the plain distance.5.000000000000 against 5.000000000000—the boundary: the plain distance is the case where the covariance was chosen to be the identity.

Observation 3.5

So the plain distance is a choice as well.Not the absence of a choice—the identity was chosen.

Observation 3.6

The three values span from 2.16 to 5.00.The same two points—a spread of 2.3, so “far” or “near” depends on which is used.

4. The Separator

The separator: has the choice vanished, or moved.

Observation 4.1

Vanished, and nothing should move the value.Neither units nor covariance—but the covariance moved it.

Observation 4.2

Moved, and what one varies changes place.From units to covariance—and indeed units left it fixed while covariance did not.

Observation 4.3

The two are distinct roots by the criterion of Paper 300.Both may be called “free of arbitrariness”—one meaning that no choice exists, the other that the choice sits elsewhere.

Observation 4.4

It stands on the boundary Paper 95 drew between convention and fact. A covariance is measured—so it looks like fact, yet which sample it is measured from and which population is assumed are choices, and that part remains on the convention side.

Observation 4.5

It has the shape of Paper 505. Three spheres of influence, with exponents set by what was balanced against—here, as many “distances” as there are covariances, with values set by what the discount is measured against.

Observation 4.6

This is one of the separators of Paper 481. When “this distance does not depend on units” is questioned—doubting the units or doubting the choice of covariance sends the repair elsewhere.

Observation 4.7

To be honest—which covariance is right is not said. Whether to measure it from a sample or take it from prior knowledge—what is shown reaches as far as: replacing it changes the value. Noting that a choice is required is not the same as saying the choice is wrong.

5. Roll Call and Closing

1. Plain distance 5.000000000000 (Computation 2.1): a three-four triangle.
2. Through the covariance, 4.123105626 (Observation 2.3): the separation discounted.
3. Units $\times 100$ give 300.026665 (Computation 2.4): a sixtyfold move.
4. The covariance-based value does not move (Observation 2.6): the core.
5. Swapping axes gives 3.282952601 (Computation 3.1): the points unchanged.
6. Correlation gives 2.157439560 (Observation 3.3): the off-diagonal at work.
7. The identity gives 5.000000000000 (Observation 3.4): the plain distance returns.
8. A spread of 2.3 (Observation 3.6): “far” or “near” changes.

9. The separator is vanished or moved (Observation 4.3): **distinct roots by Paper 300.**

10. No covariance is preferred (Observation 4.7): **only that a choice is required.**

Closing: A distance measured through a covariance is said to be free of the choice of units—the same value in metres or kilometres, the arbitrariness gone. But Σ appears in the expression, and which covariance to use is itself a choice. Changing the units does not move it—for $x = (0, 0)$ and $y = (3, 4)$, the plain distance is **5.000000000000** and the distance through $\text{diag}(9, 1)$ is **4.123105626**, the first axis varying more so that separation along it is discounted. Multiplying the first axis by 100 moves the plain distance from 5.000000 to **300.026665**, a sixtyfold change on a change of units alone. The covariance-based distance stays at **4.123105625618** without moving a digit, since the covariance scales by the same factor and cancels against its inverse. So far “free of units” causes no trouble, and on this operation alone the arbitrariness looks gone. Replacing the covariance does move it—the same two points give **3.282952601** through $\text{diag}(1, 9)$ and **2.157439560** through a correlated covariance. Swapping the axes alone changes the value, with the points unchanged. Correlation changes it further, the off-diagonal term tying the two separations together. The identity returns **5.000000000000**, the plain distance: the boundary, where the plain distance turns out to be the case of choosing the identity. So the plain distance is a choice as well—not the absence of a choice, but the identity chosen. The three values span from 2.16 to 5.00, a spread of 2.3, so “far” or “near” depends on which is used. One thing separates them—has the choice vanished, or moved. Vanished, and nothing should move the value—but the covariance moved it. Moved, and what one varies changes place, from units to covariance; and indeed units left it fixed while covariance did not. The two are distinct roots by the criterion of Paper 300: both may be called “free of arbitrariness”, one meaning that no choice exists and the other that the choice sits elsewhere. It stands on the boundary Paper 95 drew between convention and fact: a covariance is measured and so looks like fact, yet which sample and which population are choices, and that part remains on the convention side. It has the shape of Paper 505, where three spheres of influence had exponents set by what was balanced against; here there are as many distances as covariances. This is one of the separators of Paper 481: when “this distance does not depend on units” is questioned, doubting the units or doubting the covariance sends the repair elsewhere. To be honest—which covariance is right is not said. Whether to measure it from a sample or take it from prior knowledge lies outside the subject, and what is shown reaches as far as: replacing it changes the value. Noting that a choice is required is not the same as saying the choice is wrong.

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