

Longitudinal and Transverse Belong to the Field; x and y Belong to the Coordinates

— rotating the coordinates leaves the longitudinal share at **0.647222222222** without moving a digit, while the x share moves with every rotation — both are called “splitting the field in two”, and only one of them belongs to the field — [Paper 517]

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— *From thinking to syncing.* —

Abstract

There are many ways to split a vector field in two. What this paper shows is that which of them belongs to the field and which to the coordinates can be measured by rotating the coordinates. No new theorem or law is claimed. First, the longitudinal share is **0.647222222222** and the transverse share **0.352777777778**, and summing the two independently gives **1.000000000000** (Section 2). Second, splitting the same field into x and y components gives **0.384259259259**—both may be called “a split of the field in two” (Section 2). Third, this is the core of the paper. Rotating the coordinates by 30, 45 and 90 degrees leaves the longitudinal share at **0.647222222222** while the x share moves to **0.558401558841**, **0.634259259259** and **0.615740740741** (Section 3). Fourth, without a rotation the x share also stays at **0.384259259259**—the boundary (Section 3). Fifth, the three properties of the decomposition (the parts sum back, the parallel part has zero cross product, the perpendicular part has zero dot product) hold to the digit **0** for all three modes (Section 2). Sixth, the separator is the operation of rotating the coordinates—what one varies decides where a split lives (Section 4). **Keywords:** Helmholtz decomposition, vector field, rotational invariance, longitudinal and transverse parts, coordinate system.

Scope of this paper (scope note): No new theorem or law is claimed—the Helmholtz decomposition and its rotational invariance are standard. Uniqueness is not claimed—the decomposition becomes unique once boundary conditions are fixed, and this paper works only in a periodic wavenumber space. Domain effects are not entered—on a bounded domain a harmonic part appears and the split becomes threefold. This paper does not go there. Three dimensions are not treated—only three modes in two dimensions, and how the transverse degrees of freedom grow with dimension is not the subject. No physical meaning is assigned—correspondences such as longitudinal for sound and transverse for light are case by case; this paper looks only at the numbers. Transformations other than rotation are not examined—what survives stretching or shear is not treated. Relation to earlier papers: Paper 347 showed that “cannot be constructed” is a property of the tools, not of the figure—this paper runs the other way, writing one case in which a property belongs to the object and not to the tools. Paper 516 showed that freedom from units had merely handed the choice to the covariance—there the choice moved, here it genuinely vanishes; placing the two side by side shows the boundary. Paper 130 showed that Helmholtz’s three theorems are three claims at different prices—the decomposition here is a different Helmholtz bearing the same name, kept apart by number so the two are not confused. Paper 299 showed that a drum has no pitch because of its boundary—the restriction to the periodic case here is precisely to avoid stepping on that. Paper 300 showed that whether two things share a root is decidable—the single phrase “splitting the field in two” names two things with distinct roots. Paper 481 showed that without a separator the blame for a refutation has no address—this paper writes one such separator. What is added is applying the two splits to the same field and reporting the shares, showing by numbers that a rotation moves only one of them, summing the transverse part independently rather than subtracting it from the longitudinal one, checking the three properties of the decomposition mode by mode, and placing the separator at the operation of rotating the coordinates.

1. Notation and the Question

Setting: Place on the plane a field made of three superposed waves. Each wave is fixed by a wavenumber k and an amplitude vector a . The ones used are

$$(k, a) = ((1, 0), (0.7, 0.4)), \quad ((0, 1), (-0.3, 0.9)), \quad ((1, 2), (0.5, -0.6)).$$

The first split (longitudinal and transverse): For each wave, split the amplitude into the part parallel to the wavenumber and the part perpendicular to it. With $\hat{k} = k/|k|$,

$$a_{\parallel} = (a \cdot \hat{k}) \hat{k}, \quad a_{\perp} = a - a_{\parallel}.$$

This is the Helmholtz decomposition seen in wavenumber space.

The second split (x and y): Split the same amplitude into its first and second components, $a = (a_x, a_y)$.

Observation 1.1

Both may be called “a split of the field in two”. The parts sum back and the sum of squares is preserved—in form they are indistinguishable.

Question: Do the two splits stand on equal footing?

2. Applying Both Splits to the Same Field

Computation 2.1: The longitudinal and transverse parts of each mode.

Wavenumber k and amplitude a	Longitudinal $a_{\parallel}$	Transverse $a_{\perp}$
$k = (1, 0), a = (0.7, 0.4)$	(0.700000000, 0.000000000)	(0.000000000, 0.400000000)
$k = (0, 1), a = (-0.3, 0.9)$	(0.000000000, 0.900000000)	(-0.300000000, 0.000000000)
$k = (1, 2), a = (0.5, -0.6)$	(-0.140000000, -0.280000000)	(0.640000000, -0.320000000)

Observation 2.2

The first two can be checked by eye. When k lies along the x axis the longitudinal part is the x component itself, and when k lies along y it is the y component—the independently checkable cases were placed first.

Observation 2.3

The third cannot be read off. Since $k = (1, 2)$ lies along no axis, the longitudinal part spreads over both components—from here the computation carries the weight.

Check 2.4: That the decomposition is one is confirmed by three properties: the parts sum back, the parallel part really is parallel to k (cross product zero), the perpendicular part really is perpendicular to k (dot product zero). For all three modes the difference, the cross product and the dot product are 0.000×10^0 .

Computation 2.5: The energy shares.

Split	Share
Longitudinal	0.647222222222
Transverse	0.352777777778
$\mathfrak{x}$ component	0.384259259259

Observation 2.6

Summed independently, longitudinal and transverse give **1.000000000000**.The transverse share was not obtained as one minus the longitudinal one but by summing the squares of the perpendicular parts separately—a number made by subtraction is no check on a sum.

Observation 2.7

So far the two splits differ in nothing.Each returns a single number between 0 and 1, and for each the remainder sums to 1—one number for one field, the same shape in both cases.

3. A Rotation Moves Only One of Them

Computation 3.1: Rotate both wavenumbers and amplitudes by the same angle and measure the two shares again.

Angle	Longitudinal share	$\mathfrak{x}$ share
0 degrees	0.647222222222	0.384259259259
30 degrees	0.647222222222	0.558401558841
45 degrees	0.647222222222	0.634259259259
90 degrees	0.647222222222	0.615740740741

Observation 3.2

The longitudinal share does not move in twelve digits. At 0 and at 90 degrees, **0.647222222222** against **0.647222222222**—agreement inside 10^{-12} , not merely to the rounding.

Observation 3.3

The x share moves a great deal. From **0.384259259259** to **0.634259259259**, a span of **0.25**—nothing about the field was changed; only the naming of the coordinates.

Observation 3.4

Without a rotation the x share does not move either. It stays at **0.384259259259**—the boundary. The check does not say “it always moves”; it says the movement follows the rotation.

Observation 3.5

Note that 90 degrees does not restore the original. The value **0.615740740741** is not the **0.384259259259** of 0 degrees—a quarter turn carries the x axis to where y was, so the x share becomes the former y share; the two sum to **1.000000000000**.

Observation 3.6

Why only one of them holds still can be read from the expressions. The longitudinal share is built only out of the dot product $a \cdot \hat{k}$, and a dot product survives rotation—the x share contains the first component of a , which is a coordinate's name.

4. The Separator

Observation 4.1

The separator is the operation of rotating the coordinates. If the value moves it is a property of the coordinates; if it does not, a property of the field—one operation parts the two.

Observation 4.2

Replace that operation and the separator stops working. Scaling the whole field by a constant moves neither share—an operation that does not part the two is no separator.

Observation 4.3

“Splitting the field in two” has distinct roots. By the criterion of Paper 300, the longitudinal-transverse split and the xy split share a form and not an origin—sameness of form is no evidence of a shared root.

Observation 4.4

Placed beside Paper 516, three cases appear. The choice vanishes (the longitudinal split here), the choice moves (the covariance of Paper 516), the choice remains (the xy split here)—when something is said to be free of arbitrariness, these three must be told apart.

Observation 4.5

The direction is opposite to Paper 347. There, what was taken for a property of the figure turned out to belong to the tools—here, what might be taken for an artefact of coordinates belongs to the field. Two answers to one question.

Observation 4.6

In the language of Paper 481, doubting “this decomposition does not depend on coordinates” has two addresses. Doubt the definition of the decomposition, or doubt the implementation of the rotation—numbers were placed at both, so a check remains wherever the blame falls.

Observation 4.7

Which split is correct is not said. There is no end of settings in which splitting into x and y is the useful thing to do—what can be said is only that such a split need not be saying anything about the field.

5. Roll Call and Closing

1. Longitudinal share **0.647222222222** (Computation 2.5): for a field of three waves.
2. Transverse **0.352777777778**, summing independently to **1.000000000000** (Observation 2.6): not made by subtraction.
3. The x share **0.384259259259** (Computation 2.5): no difference so far.
4. Under rotation the longitudinal share holds at **0.647222222222** (Observation 3.2): the core.
5. The x share runs to **0.558401558841**, **0.634259259259**, **0.615740740741** (Computation 3.1): a span of **0.25**.
6. Without a rotation neither moves (Observation 3.4): the boundary.
7. At 90 degrees the x share becomes the former y share (Observation 3.5): summing to **1.000000000000**.
8. The reason is that a dot product survives rotation (Observation 3.6): it sits in the expressions.
9. The separator is the rotation (Observation 4.1): scaling does not serve.
10. Which split is correct is not said (Observation 4.7): only that one need not speak of the field.

Closing: There are many ways to split a vector field in two, and all of them have the same form: the parts sum back. Sameness of form does not mean sameness of address—for a field of three superposed waves the longitudinal share is **0.647222222222** and the transverse share **0.352777777778**, and summing the transverse part independently rather than subtracting it gives **1.000000000000**. Splitting the same field into x and y gives **0.384259259259**, and to this point the two splits differ in nothing. Each returns one number, and for each the remainder sums to one. A rotation moves only one of them—through 30, 45 and 90 degrees the longitudinal share holds at **0.647222222222** in all twelve digits, while the x share runs to **0.558401558841**, **0.634259259259** and

0.615740740741, a span of **0.25**. Nothing about the field was changed; only the naming of the coordinates. Without a rotation the x share also stays at **0.384259259259**, so the check is not saying that it always moves. A quarter turn does not restore the original because the x share becomes the former y share, the two summing to **1.000000000000**. The reason sits in the expressions: the longitudinal share is built from a dot product, and a dot product survives rotation, whereas the x share carries a coordinate's name. One operation parts the two—if the value moves under rotation it belongs to the coordinates, and if it does not, to the field. Scaling the field by a constant moves neither, so the separator is the rotation and not the scaling. By the criterion of Paper 300 the two splits share a form and not an origin. Set beside Paper 516, three cases appear: the choice vanishes, the choice moves to the covariance, the choice simply remains. The direction is opposite to Paper 347, where what was taken for a property of the figure belonged to the tools; here what might be taken for an artefact of coordinates belongs to the field. In the language of Paper 481, doubting the invariance can fall on the definition of the decomposition or on the implementation of the rotation. To be honest—only a periodic wavenumber space was treated. On a bounded domain a harmonic part appears, the split becomes threefold, and it is not even unique until boundary conditions are fixed. As Paper 299 showed, a boundary tells as much as an equation. What this paper can say reaches as far as what failed to move when the boundary was set aside and only the rotation was varied.

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