Zeno’s Four Paradoxes Do Not Share a Single Resolution

— the dichotomy falls to a convergent series, Achilles to an infinite sum, the arrow to the limit of a difference quotient and the stadium to a discrete space — four paradoxes, four different pieces of mathematics — [Paper 519]

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— *From thinking to syncing.* —

Abstract

Zeno’s paradoxes are often said, all together, to have been settled by the calculus.What this paper shows is that the four fall to four different tools, and that one of them does not fall to the calculus at all.No new theorem or law is claimed. First, the dichotomy falls to the convergence of a geometric series—at fifty terms the sum is $\mathbf{0.999999999999999}$, short of one by $\mathbf{8.882 \times 10^{-16}}$ (Section 2). **Second, Achilles falls to an infinite sum agreeing with a closed form—twenty stages give $\mathbf{11.111111111111}$ seconds, within $\mathbf{1.776 \times 10^{-15}}$ of $d/(v_1 - v_2)$** (Section 2). **Third, the arrow falls to the limit of a difference quotient—it converges to $\mathbf{20.000000000000}$, yet at $h = 10^{-7}$ it worsens to $\mathbf{19.999999985032}$** (Section 3). **Fourth, this is the core of the paper.The stadium does not fall to the calculus—on a lattice the motion is $\mathbf{2}$ cells relative to the opposing row and $\mathbf{1}$ relative to the stationary one, so half a tick demands $\mathbf{0.5}$ of a cell, which the lattice cannot write** (Section 3). **Fifth, the core of the arrow is that a position at one instant does not fix a velocity—two motions passing through $\mathbf{20.000000}$ at $t = 2$ have velocities $\mathbf{20.000000}$ and $\mathbf{3.000000}$** (Section 3). **Sixth, the separator is what one must assume to bring the paradox back—that the four are brought back in different ways is the evidence that they are four** (Section 4). **Keywords**: Zeno’s paradoxes, Achilles and the tortoise, geometric series, philosophy of motion, discrete space.

Scope of this paper (scope note): **No new theorem or law is claimed—the sum of a geometric series, the definition of the derivative and relative motion on a discrete space are all standard.Zeno's text is not interpreted—it survives only through Aristotle and Simplicius, and this paper treats only the four forms as handed down. What Zeno meant is not the subject.No philosophical verdict is reached—whether infinite division is possible is not entered; only which tool bites, mathematically.Nonstandard analysis is not entered—formulations treating infinitesimals as objects are not the subject.No claim is made that space is discrete—it is not said that space is in fact a lattice, only what follows if one assumes so.Variants of the fourth paradox are not weighed—the stadium comes down in several forms, and only the reading as relative motion is treated.Relation to earlier papers**: **Paper 213 showed that "coming back" has distinct roots—this paper shows the same of "resolved", putting one observation under a different verb.Paper 300 showed that whether two things share a root is decidable—the four paradoxes are judged distinct by the differences in how each is brought back.Paper 517 separated two splits by the single operation of rotating the coordinates—the separator here is likewise an operation, and four of them are needed.Paper 518 showed that a disagreement between priors expires—there one expression's span narrowed; here four expressions never merge.Paper 481 showed that without a separator the blame for a refutation has no address—doubting "Zeno was resolved" lands in a different place depending on which of the four is meant.**What is added is **giving each of the four its own numbers**, **matching the infinite sum for Achilles against a closed form**, **measuring how a difference quotient worsens when h is made too small**, **placing two motions that share a position and differ in velocity**, and **showing from the discrete side that only the stadium resists the calculus**.

1. Notation and the Question

Setting: **The four paradoxes are set down as they have come to us.**

The dichotomy: **To reach a destination one must first cover half, then half of what remains, through infinitely many stages.**

Achilles: **By the time Achilles reaches where the tortoise was, the tortoise has moved on.**

The arrow: **At every instant the arrow is at rest. How does a succession of rests make a motion?**

The stadium: **Two rows moving in opposite directions move faster relative to each other than to a row at rest.**

Observation 1.1

The four are commonly gathered under "the calculus settled them".One sentence: an infinite series can have a finite sum—whether that sentence bites equally on all four is what this paper checks.

Question: Does the word “resolved” name the same thing in all four cases?

2. The Dichotomy and Achilles — Two That a Series Settles

Computation 2.1: Sum the stages of the dichotomy.

Terms	Sum	Short of 1 by
10	**0.999023437500000**	$\mathbf{9.766 \times 10^{-4}}$
30	**0.999999999068677**	$\mathbf{9.313 \times 10^{-10}}$
50	**0.999999999999999**	$\mathbf{8.882 \times 10^{-16}}$

> **Observation 2.2**
>
> The stages are infinite in number and the sum is finite.At fifty terms the gap closes to $\mathbf{8.882 \times 10^{-16}}$—at ten terms it still stands at $\mathbf{9.766 \times 10^{-4}}$, so the check responds to the number of terms.

Computation 2.3: Put numbers on Achilles.Take his speed $v_1 = 10$, the tortoise’s $v_2 = 1$ and the head start $d = 100$ .The closed form is $d/(v_1 - v_2) = \mathbf{11.111111111111}$ seconds.

Stages summed	Time	Difference from the closed form
5	**11.111000000000**	$\mathbf{1.111 \times 10^{-4}}$
20	**11.111111111111**	$\mathbf{1.776 \times 10^{-15}}$
60	**11.111111111111**	$\mathbf{1.776 \times 10^{-15}}$

> **Observation 2.4**
>
> The infinite sum agrees with the overtaking formula.$\mathbf{11.111111111111}$ against $\mathbf{11.111111111111}$—the paradox says the stages are infinite; the sum of infinitely many stages equals the value a finite expression gives.

Check 2.5: Confirm from the side of distance.Achilles has run $v_1 t = 111.111111111$ and the tortoise stands at $d + v_2 t = 111.111111111$—an answer got from an equation in time, checked against a different quantity.

> **Observation 2.6**
>
> These two fall to the same tool.Both to the convergence of a geometric series—counting the dichotomy and Achilles separately means little at this stage.

> **Observation 2.7**
>
> Yet sixty stages do no better than twenty.Both stand at 1.776×10^{-15}—by twenty stages the floating-point step has been reached, and further terms bring no gain. Mathematical convergence and computational convergence stop in different places.

3. The Arrow and the Stadium — Two That Need Other Tools

Computation 3.1: Take the position to be $5t^2$ and measure the difference quotient at $t = 2$.

Step h	Difference quotient
10^{-1}	20.000000000000
10^{-3}	19.999999999998
10^{-5}	20.000000000131
10^{-7}	19.999999985032

> **Observation 3.2**
>
> It converges to the closed form $10t = 20.000000000000$.Velocity is defined as a limit, not by asking whether the object is at rest at a point—the tool here is the derivative, not a series.

Observation 3.3

Making h too small makes matters worse.At $h = 10^{-7}$ the value is 19.999999985032—the truncation error falls while cancellation in the subtraction grows. A limit is an operation in mathematics, not a procedure in arithmetic.

Observation 3.4

At $h = 0$ both numerator and denominator vanish.There is no value—which is why the detour through a limit is needed. This is the point the paradox presses.

Computation 3.5: Show with two motions that a position at one instant does not fix a velocity.

Motion	Position at $t = 2$	Velocity at $t = 2$
$5t^2$	20.000000	20.000000
$20 + 3(t - 2)$	20.000000	3.000000

Observation 3.6

The positions agree and the velocities do not.20.000000 against 20.000000, and 20.000000 against 3.000000—Zeno is right that at a single instant nothing is moving: from one point no velocity can be written. The error lies in the step from there to the absence of motion.

Computation 3.7: Place the stadium on a lattice.In one tick a row moves 1 cell relative to the stationary row and 2 cells relative to the row coming the other way.

Observation 3.8

Half a tick demands an unwritable value.Relative to the opposing row it is 1.0 cell, an integer; relative to the stationary row it is 0.5 of a cell—which cannot stand with the assumption that a cell is the smallest unit.

Observation 3.9

Make the space continuous and 0.5 can be written.The other side of the boundary—what settles this paradox is neither a series nor a derivative but taking space to be continuous, or rewriting the convention of simultaneity.

Observation 3.10

So the fourth paradox does not fall to the calculus.Series and derivatives are both written over a continuous space—and that premise is what the stadium is questioning.

4. The Separator

Observation 4.1

The separator is what one must assume to bring the paradox back.The dichotomy and Achilles return if the series diverges—the arrow returns if limits are not granted; the stadium returns if space is taken to be discrete.

Observation 4.2

That they return by different routes is the evidence of distinct roots.By the criterion of Paper 300, the four gather under the phrase “a difficulty about the infinite” while what one varies differs—that a single phrase covers them is no evidence of a shared root.

Observation 4.3

Two of the four do share a root.The dichotomy and Achilles return under the same operation—the claim is not that all four differ, but that they part into two, one and one.

Observation 4.4

The shape is that of Paper 213.There the verb “to come back” had distinct roots—here it is the verb “to resolve”. Taking a verb as the subject brings the same structure out again.

Observation 4.5

It stands opposite Paper 518.There the disagreement among four positions narrowed as $1/(n+2)$—here the four never merge. Some disagreements end and some do not.

Observation 4.6

In the language of Paper 481, doubting “Zeno was resolved” has at least three addresses.The convergence of the series, the existence of the limit, the continuity of space—and which one is doubted changes where a counterexample would be sought.

Observation 4.7

What Zeno meant is not said.The text does not survive, and only the transmitted forms are at hand—what can be said is that against those four forms, the tools that bite come apart into four.

5. Roll Call and Closing

1. **The dichotomy is short by 8.882×10^{-16} at fifty terms** (Computation 2.1): **a geometric series.**
2. **Achilles at 11.111111111111 seconds** (Computation 2.3): **sum equals closed form.**
3. **Distance agrees at 111.111111111** (Check 2.5): **checked in another quantity.**
4. **Sixty stages do no better than twenty** (Observation 2.7): **arithmetic stops first.**
5. **The arrow converges to 20.000000000000** (Observation 3.2): **the tool is the derivative.**
6. **At $h = 10^{-7}$ it worsens to 19.999999985032** (Observation 3.3): **cancellation wins.**
7. **Same position 20.000000, velocities 20.000000 and 3.000000** (Computation 3.5): **one point fixes nothing.**
8. **The stadium gives 2 against 1, and 0.5 at half a tick** (Observation 3.8): **the core.**
9. **The separator is what brings each back** (Observation 4.1): **different for each.**

10. **They part into two, one and one** (Observation 4.3): **not into four.**

Closing: **Zeno's four paradoxes are commonly gathered under the claim that the calculus settled them.A series settles two—the dichotomy falls to the convergence of a geometric series, whose fifty-term sum is 0.999999999999999, short of one by 8.882×10^{-16}. At ten terms the gap is still 9.766×10^{-4}, so the check responds to the number of terms. Achilles falls because the infinitely many times taken to reach where the tortoise was sum to within 1.776×10^{-15} of the closed form $d/(v_1 - v_2) = 11.111111111111$ seconds. The same answer was checked in distance, 111.111111111 against 111.111111111. Yet sixty stages do no better than twenty: the floating-point step is reached at twenty, and mathematical convergence and computational convergence stop in different places.The third needs another tool—the arrow falls to the limit of a difference quotient, converging to 20.000000000000. Making h too small makes matters worse: at $h = 10^{-7}$ the value is 19.999999985032, the truncation error falling while cancellation grows, since a limit is an operation in mathematics and not a procedure in arithmetic. At $h = 0$ both numerator and denominator vanish and there is no value. Two motions through the same position 20.000000 at $t = 2$ carry velocities 20.000000 and 3.000000: Zeno is right that at a single instant nothing moves, and the error lies in the step from there to the absence of motion.The fourth does not fall to the calculus—on a lattice, one tick carries a row 1 cell relative to the stationary row and 2 relative to the row coming the other way. At half a tick the opposing figure is 1.0 cell and remains an integer, while the stationary one demands 0.5 of a cell, which cannot stand with a cell being the smallest unit. Make the space continuous and 0.5 can be written; what settles this paradox is neither series nor derivative but the continuity of space. Series and derivatives are both written over a continuous space, and that premise is what the stadium questions.The separator is the route back—let the series diverge and two return; refuse limits and the third returns; take space to be discrete and the fourth returns. By the criterion of Paper 300 they gather under "a difficulty about the infinite" while what one varies differs. The claim is not that all four differ, but that they part into two, one and one. Paper 213 found distinct roots under the verb "to come back"; here they sit under the verb "to resolve". Opposite Paper 518, where a disagreement narrowed as $1/(n+2)$, these four never merge. In the language of Paper 481, doubting "Zeno was resolved" lands on the convergence of the series, the existence of the limit, or the continuity of space.To be honest—what Zeno meant has not been said. The text does not survive, and only the transmission through Aristotle and Simplicius is at hand. The stadium comes down in several forms and only the reading as relative motion was treated. Nor is it claimed that space is in fact a lattice. What can be said reaches as far as: against the four transmitted forms, the tools that bite come apart into four.**

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