The Same “Unprovable” Splits into One from Diagonalisation and One from Growth

— exhausting $R(5,5)$ faces 2^{903} colourings, finite yet out of reach, and the Goodstein sequence from $n = 4$ ends only after $3 \cdot 2^{402653211} - 2$ steps, yet its ordinal decreases strictly at every step — [Paper 525]

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— From thinking to syncing. —

Abstract

The same phrase, “unprovable in Peano arithmetic”, is applied to Godel’s sentence and to Goodstein’s theorem.What this paper shows is that the two come from different roots, one from self-reference and one from the speed of growth.No new theorem or law is claimed. First, $R(3,3) = 6$ is settled by exhaustion——K_5 admits a colouring with no monochromatic triangle, and none of the 32768 colourings of K_6 does (Section 2). **Second, the same finite question for $R(5,5)$ is out of reach——exhaustion faces 2^{903} colourings, so being finite and being computable are different** (Section 2). **Third, the Goodstein sequence from $n = 3$ ends in the 6 terms $3, 3, 3, 2, 1, 0$ ——five steps, checkable on the spot** (Section 3). **Fourth, and this is the core.From $n = 4$ the integers keep growing while the ordinal shrinks at every step——the sequence climbs $4, 26, 41, 60, 83, 109, 139, 173, 211, 253, 299, 348$ monotonically, while the ordinal read off before the base is raised decreases strictly each time** (Section 3). **Fifth, termination is provable while the time to terminate is not writable with finite tools——from $n = 4$ it reaches 0 after $3 \cdot 2^{402653211} - 2$ steps** (Section 3). **Sixth, the separator is what is decreasing——Godel’s sentence needs a diagonal, Goodstein needs ordinals up to ε_0** (Section 4). **Keywords**: incompleteness theorem, Paris-Harrington theorem, Goodstein’s theorem, Peano arithmetic, Ramsey theory.

Scope of this paper (scope note): **No new theorem or law is claimed——Godel's incompleteness theorems, the Paris--Harrington theorem and the Kirby--Paris result on Goodstein sequences are all standard.No incompleteness theorem is proved——what is done is to set two already-proved independence results side by side.“The first example without self-reference” is not asserted——it is often so called, but what counts as first depends on the definition.Ordinal analysis is not entered——that ε_0 is the proof-theoretic ordinal is used, but the argument fixing it is not discussed.No position in the foundations of mathematics is taken——formalism against realism is not the subject.No complexity classification is claimed——2^{903} is not called “unsolvable”; only that exhaustion does not reach it.Relation to earlier papers**: **Paper 189 showed that “unprovable” is not a property of a sentence——this paper continues it by showing that one such phrase has two origins.Paper 259 showed there are two roads to independence, one narrowing the universe and one widening it——that paper divided the methods; this one divides the sentences.Paper 302 showed that “it must appear” does not say when——the 2^{903} here is the concrete size of that silence.Paper 300 showed that sharing a root is decidable——the two independence results are separate roots under the criterion of route.**What is added is **settling $R(3,3)=6$ by explicit exhaustion**, counting the exhaustion for **$R(5,5)$ as 2^{903}**, **running the Goodstein sequences to obtain the six terms at $n=3$ and the climb at $n=4$**, **checking mechanically that the ordinal falls at every step while the integers rise**, and **placing the separator on what is decreasing**.

1. Notation and the Question

Setting: **Ramsey's theorem says that colouring a large enough complete graph in two colours forces a monochromatic complete subgraph.The least such size is $R(s,t)$.**

Goodstein sequences: **write n in hereditary base 2, raise the base to 3, subtract 1; then raise base 3 to 4, subtract 1; and so on.**

> **Observation 1.1**
>
> **both are statements about finite objects.No infinite sets, no real numbers, no self-reference——and yet the strengthened versions are unprovable in Peano arithmetic.**

> **Observation 1.2**
>
> **this does not look like Godel's sentence.Godel's sentence speaks about itself, saying it cannot be proved——Goodstein's theorem speaks only about a sequence, mentioning neither proofs nor arithmetic.**

Question: **is it the same “unprovable”?**

2. Ramsey Numbers Yield to Exhaustion

Computation 2.1: examining every two-colouring of K_5 and K_6 for the presence of a colouring free of monochromatic triangles.

graph	colourings examined	triangle-free colouring
K_5	$2^{10} = 1024$	exists
K_6	$2^{15} = \mathbf{32768}$	none

> **Observation 2.2**
>
> $R(3,3) = 6$ was settled here.Not existence but the value itself——a statement about finite objects is, in principle, settled by exhaustion.

> **Observation 2.3**
>
> examining both sides is what makes it work.K_5 escapes and K_6 cannot——one side alone is indistinguishable from “not found yet”.

Computation 2.4: turning the same procedure on $R(5,5)$. Since $R(5,5) \le 43$ is known, K_{43} has $\binom{43}{2} = 903$ edges and 2^{903} colourings.

> **Observation 2.5**
>
> finite and computable are different.2^{903} is finite and beyond any machine in this universe——the silence recorded in Paper 302 now has a number attached: 903.

> **Observation 2.6**
>
> no unprovability has appeared so far.Everything so far is merely “too large to check”——Paris and Harrington added one condition, relative largeness, and that alone changes the situation.

3. Goodstein: Integers Rise, Ordinals Fall

Computation 3.1: **running the sequence from $n = 3$.**

quantity	value
sequence	**3, 3, 3, 2, 1, 0**
terms	**6**
steps	**5**

> **Observation 3.2**
>
> **for a small seed the ending is visible.Zero after six terms——no need to trust the theorem; running it suffices.**

Computation 3.3: **running from $n = 4$ and listing the first 12 terms.**

4, 26, 41, 60, 83, 109, 139, 173, 211, 253, 299, 348

> **Observation 3.4**
>
> **it climbs monotonically.Not once does it fall over twelve terms——growth is all that is visible, and there is no sign of an ending.**

Computation 3.5: **reading the same terms as hereditary expressions before the base is raised, and comparing them as ordinals, with the base in the role of ω, shows a strict decrease at every one of the twelve steps.**

> **Observation 3.6**
>
> **this is the core.At one and the same step the integer rises and the ordinal falls——two quantities move in opposite directions, and only the falling one reveals termination.**

Observation 3.7

terminating and knowing when are different.From $n = 4$ the sequence reaches 0 after $3 \cdot 2^{402653211} - 2$ steps ——termination is provable while the length is not writable with finite tools.

Observation 3.8

what Peano arithmetic lacks is that ordinal.Transfinite induction up to ε_0 would prove termination, and Peano arithmetic does not have it——what is missing is not logical strength but the height of the ordinal guaranteeing that descending chains stop.

4. Two Kinds of “Unprovable”

Observation 4.1

the separator is what is decreasing.Godel’s sentence is unprovable because a sentence can be made to point at itself——Goodstein is unprovable because the growth outruns the system’s ordinal.

Observation 4.2

separate roots under the route criterion of Paper 300.One goes by diagonalisation, the other by ordinal analysis——they arrive at the same conclusion along roads that are not shared.

Observation 4.3

there is a mark that tells them apart.Independence coming from growth disappears when the statement is weakened——drop “relatively large” from Paris--Harrington and ordinary finite Ramsey remains, provable in Peano arithmetic. Godel’s sentence cannot be weakened.

Observation 4.4

this continues Paper 189.There, “unprovable” was shown to be a property of a pair, sentence and system——here, even one pair admits two origins.

Observation 4.5

the orientation differs from Paper 259.That paper divided the two methods of proving independence——this one divides the sentences. Classifying methods and classifying objects are different tasks.

Observation 4.6

what is not said.“The first example without self-reference” is not asserted——it is often so called, but what counts as first depends on the definition.

5. Roll Call and Closing

1. K_5 **has an escape** (Computation 2.1): 1024 **colourings examined.**
2. **none of the** 32768 **colourings of** K_6 **does** (Computation 2.1): **hence** $R(3,3) = 6$**.**
3. $R(5,5)$ **needs** 2^{903} (Computation 2.4): **finite and unreachable.**
4. $n = 3$ **gives** $3, 3, 3, 2, 1, 0$ (Computation 3.1): **six terms, five steps.**
5. $n = 4$ **gives** $4, 26, 41, \ldots, 348$ (Computation 3.3): **monotone climb.**
6. **the ordinal falls strictly at every step** (Computation 3.5): **the core.**
7. $n = 4$ **halts after** $3 \cdot 2^{402653211} - 2$ **steps** (Observation 3.7): **halting is not length.**
8. **what is missing is induction to** ε_0 (Observation 3.8): **not logical strength.**
9. **one independence vanishes on weakening** (Observation 4.3): **the distinguishing mark.**
10. **the separator is what decreases** (Observation 4.1): **a diagonal or an ordinal.**
11. **“first example” is not asserted** (Observation 4.6): **it depends on the definition.**

Closing: **The same phrase, “unprovable in Peano arithmetic”, is applied to Godel’s sentence and to Goodstein’s theorem.Finite objects yield, in principle, to exhaustion——among the** 1024 **colourings of** K_5 **there is one with no monochromatic triangle, and among the** 32768 **colourings of** K_6 **there is none, so** $R(3,3) = 6$ **is settled here as a**

value and not merely as an existence claim. Examining both sides is what makes it work, since one side alone cannot be told from "not found yet". Turning the same procedure on $R(5,5)$ meets 2^{903} colourings, so finite and computable part company, and the silence of Paper 302 acquires the number 903.In Goodstein's sequence two quantities move opposite ways at one step——from $n = 3$ the sequence is $3, 3, 3, 2, 1, 0$, six terms and five steps, and running it is enough. From $n = 4$ it climbs $4, 26, 41, 60, 83, 109, 139, 173, 211, 253, 299, 348$ without once falling, so nothing visible suggests an ending. Read as ordinals before the base is raised, however, the same twelve steps decrease strictly every time. Termination is provable, and yet from $n = 4$ it arrives only after $3 \cdot 2^{402653211} - 2$ steps, a length no finite tool writes down.The separator is what is decreasing——Godel's sentence is unprovable because a sentence can point at itself; Goodstein is unprovable because the growth outruns the system's ordinal, and what Peano arithmetic lacks is transfinite induction to ε_0, not logical strength. There is a mark: independence born of growth vanishes when the statement is weakened, since dropping relative largeness from Paris--Harrington leaves ordinary finite Ramsey, which is provable, whereas Godel's sentence cannot be weakened. Under the route criterion of Paper 300, diagonalisation and ordinal analysis are not a shared road, so the same conclusion has two roots. Paper 189 showed "unprovable" belongs to a pair; this paper shows one pair can still have two origins.To be honest——no incompleteness theorem was proved here; two already-proved results were set side by side. Ordinal analysis was not entered, only the fact that ε_0 is the proof-theoretic ordinal was used. "The first example without self-reference" was not asserted, since what counts as first depends on the definition, and 2^{903} was not called unsolvable, only unreachable by exhaustion. What can be said is that one phrase covers two roots, that one of them disappears when the statement is weakened and the other does not, and no further.

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