

The Yamagishi Born-Rule Geometric Structure Theory

— 3rd ed.: The Exponent 2 Comes from Two Distinct Roots, and Here the 4π Survives — [Paper 53]

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— *From thinking to syncing.* —

ABSTRACT

Against the probability interpretation $P = |\psi|^2$ that Max Born laid down as an axiom in 1926, we argue from two routes **why the exponent must be 2**. First, the only power functional conserved under unitary evolution **for every initial state** is $\int |\psi|^2$ (Theorem 2.1, verified symbolically). Second, in three dimensions the condition that the probability flux through a sphere be independent of the radius, for the outgoing wave $\psi = Ae^{ikr}/r$, is $|\psi|^2 \propto r^{-2}$ — **the same algebraic condition** as the $n = 2$ uniqueness theorem of Paper 01 (Theorem 3.1). What the third edition adds is the **address** of these two 2s. **First, the two routes come from different roots**. The 2 of Theorem 2.1 is fixed by **dynamics** (unitarity); the 2 of Theorem 3.1 is fixed by **geometry** (the area of a sphere goes as r^2). Both point at $n = 2$, and **the agreement is not a coincidence but neither is it an identity** (Observation 3.4). **Second, the “alternatives” Theorem 3.1 excludes do not exist**. The state $\psi = Ae^{ikr}r^{-n/2}$ corresponding to $|\psi|^2 \propto r^{-n}$ solves the free Schrödinger equation only for $n = 2$ (verified symbolically). Paper 01's $\rho \propto r^{-n}$ is a family in which **every n is a possible distribution**; this family has no member but one. **The algebra is the same; what the algebra acts on is not** (Observation 3.5). **Third, the 4π does not vanish here**. Paper 51 showed that in the shell theorem 4π appears on both sides and cancels completely. In $\Phi = 4\pi(\hbar k/m)|A|^2$ it appears **only once**, from the area of the sphere, and there is nothing to cancel it against, so it survives. The 4π of Paper 52 was not even a solid angle. **One symbol, three fates** (Observation 3.6). This paper claims no complete derivation of the Born rule. Setting out its relation to prior work, beginning with Gleason's theorem, it argues for the structural necessity of the exponent 2 and for where that necessity comes from.

Scope of this paper: every tool used is standard (conservation of $\int |\psi|^2$ under unitarity, the probability current $\mathbf{j} = (\hbar/m)\text{Im}(\psi^*\nabla\psi)$, the outgoing spherical wave). **NO NEW PHYSICAL LAW AND NO NEW THEOREM**. What this paper does is write separately where the exponent 2 comes from. Section 5 (the universe-OS reading) is **Layer 3 (non-scientific)**, and Sections 2 to 4 depend on none of it.

1. Introduction: why the square?

The Born rule of quantum mechanics,

$$P(x) d^3x = |\psi(x)|^2 d^3x$$

is laid down as an empirical axiom in the standard formulation. The long-standing question is a single one: **why is probability the square of the amplitude? Why not the first power, or the cube?**

This paper gives two structural answers, both checkable within existing quantum mechanics and requiring no additional assumption. **And the third edition shows that the two do not become 2 for the same reason.**

2. First route: unitarity fixes the exponent

Theorem 2.1 (uniqueness of the exponent as a conserved quantity)

Under the unitary evolution $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ with $U^\dagger U = I$, the functional

$$N_n[\psi] \equiv \int |\psi(x)|^n d^3x$$

is time-invariant **for every initial state** only for $n = 2$.

Proof For $n = 2$, conservation of the inner product gives

$$N_2[\psi(t)] = \langle \psi(t) | \psi(t) \rangle = \langle \psi(0) | U^\dagger U | \psi(0) \rangle = N_2[\psi(0)]$$

so it is conserved directly by unitarity. For $n \neq 2$ a counterexample suffices. The free-particle Gaussian packet spreads monotonically in width $\sigma(t)$, and the normalised density is $|\psi|^2 = (\pi\sigma^2)^{-3/2} \exp(-r^2/\sigma^2)$. Then

$$N_n[\psi] \propto \sigma^{3(1-n/2)}$$

and for $n \neq 2$ the exponent $3(1 - n/2) \neq 0$, so N_n depends on σ . Since the packet spreads, N_n changes with time (verified symbolically: for $n = 1, 3/2, 2, 5/2, 3, 4$ the powers of σ are $3/2, 3/4, 0, -3/4, -3/2, -3$).

Corollary 2.2

The total probability must be conserved. By Theorem 2.1 the only power of the amplitude meeting that requirement is $n = 2$.

Observation 2.3 (third edition: the force of the theorem lies in the word “every”)

The strength of Theorem 2.1 rests on requiring **all** states, not **some one** state. Indeed, **in a stationary state (an energy eigenstate)** $|\psi(\mathbf{x}, t)|$ is time-independent, so N_n is conserved for every n . Looking at a single state, the exponent is not pinned down at all.

What pins it down is **the dynamics of spreading**. Because σ moves, $\sigma^{3(1-n/2)}$ moves, and only $n = 2$ survives. **The address of this 2 is dynamics, not geometry** — which is why it must be distinguished from the 2 of Section 3 (Observation 3.4).

Remark 2.4 (the range of this theorem — important)

Theorem 2.1 carries **two limitations**.

First, it is a uniqueness statement inside the family of power functionals $\int |\psi|^n d^3x$. It narrows the power **after assuming** that the probability density is a power of $|\psi|$. Functionals that are not powers of $|\psi|$ are not excluded. **Gleason's theorem is stronger than this paper precisely because it needs no such assumption** (Section 4).

Second, it does not answer why $|\psi|^2$ is a *probability*. Why a conserved quantity should be read as probability is a question of measurement theory, outside this theorem.

3. Second route: probability flux and the $n = 2$ uniqueness theorem

Theorem 3.1 (identity of the spherical probability flux with $n = 2$ uniqueness)

For the outgoing spherical wave in three dimensions,

$$\psi(\mathbf{r}) = A \frac{e^{ikr}}{r}$$

the condition that the probability flux through the sphere of radius r be independent of r is $|\psi|^2 \propto r^{-2}$, the same algebraic condition as in the $n = 2$ uniqueness theorem of Paper 01.

Proof The probability flux is not the density itself but the surface integral of the probability current

$$\mathbf{j} = \frac{\hbar}{m} \text{Im}(\psi^* \nabla \psi)$$

For the spherical wave this gives

$$j_r = \frac{\hbar}{m} \text{Im}(\psi^* \partial_r \psi) = \frac{\hbar k}{m} \cdot \frac{|A|^2}{r^2}$$

(verified symbolically). The total flux through the sphere of radius r , with the area element $dA = r^2 d\Omega$, is therefore

$$\Phi(r) = \oint_{S^2} j_r r^2 d\Omega = \frac{\hbar k}{m} \cdot \frac{|A|^2}{r^2} \cdot 4\pi r^2 = 4\pi \frac{\hbar k}{m} |A|^2 \quad (8)$$

independent of r . The essential point is that **the r^2 of the area element cancels the r^{-2} of j_r** . In this spherical wave j_r is proportional to $|\psi|^2$ with a constant of proportionality $\hbar k/m$ (the group velocity) that does not depend on r ; the argument about r -dependence therefore carries over to $|\psi|^2$ unchanged. Setting $|\psi|^2 \propto r^{-n}$ gives $j_r \propto r^{-n}$ and

$$\Phi(r) \propto 4\pi r^{2-n}$$

so Φ is independent of r if and only if $n = 2$. This is the same algebraic condition derived in Paper 01 as the condition for the shell mass rate $dm/dr = 4\pi r^{2-n}$ to be constant.

[Second-edition correction, unchanged]: The first edition wrote $\int_{S^2} |\psi|^2 r^2 d\Omega$ as “the probability flux”. **That quantity is not a flux** — its dimension is $[L^{-1}]$, not probability per time. The conclusion is unchanged, but the identification of $|\psi|^2$ with the current $\mathbf{j}$ was corrected as above.

Corollary 3.2 (correspondence of the two $n = 2$)

Theorem 3.1 shows that two statements in this corpus reduce to the same algebraic condition:

Context	Conserved quantity	Condition
Paper 01 (mass distribution)	shell mass rate dm/dr	$\rho \propto r^{-2}$
This paper (quantum probability)	flux through a sphere Φ	$ \psi ^2 \propto r^{-2}$

Both follow from the single geometric fact that the area of a sphere in three dimensions goes as r^2 .

Remark 3.3

What Theorem 3.1 shows is that **given the Born rule**, the probability density obeys an r^{-2} law in three dimensions, and that this condition is isomorphic to Paper 01's. It is not a derivation of the Born rule but a proof of **its consistency with three-dimensional geometry**.

3.1 [Third edition] The addresses of the two 2s

Observation 3.4 (the 2 of dynamics and the 2 of geometry)

Sections 2 and 3 both arrive at $n = 2$, but **what carries them there differs**.

	What fixes the 2	Dependence on dimension
Theorem 2.1 (unitarity)	dynamics : the packet spreads	none ($n = 2$ in every d)
Theorem 3.1 (flux)	geometry : the sphere's area is r^2	yes (in d dimensions, $n = d - 1$)

The decisive distinction is in the right-hand column. **The 2 of Theorem 2.1 stays 2 when the spatial dimension changes** — for a d -dimensional Gaussian packet $N_n \propto \sigma^{d(1-n/2)}$, and the exponent vanishes at $n = 2$ for every d . **The 2 of Theorem 3.1 is specific to $d = 3$** ; in d dimensions it becomes $n = d - 1$ (Papers 01 and 114).

Hence **the two agree in three dimensions only because $d - 1 = 2$ happens to hold there**. The agreement is a fact, not an identity. **In four dimensions Theorem 2.1 says $n = 2$ while Theorem 3.1 says $n = 3$, and the two part company**. This is the counterexample that separates the two roots.

Observation 3.5 (the alternatives Theorem 3.1 excludes do not exist)

Theorem 3.1 selects $n = 2$ within the family $|\psi|^2 \propto r^{-n}$. **But as free-particle states, that family has no member other than $n = 2$** .

Substituting $\psi = Ae^{ikr}r^{-a}$ into the free Schrödinger equation (radial Helmholtz reduction $u = r\psi$, $u'' + k^2u = 0$) gives

$$u'' + k^2u = r^{-a-1} (a^2 - a - 2ikr(a - 1))e^{ikr}$$

which vanishes for all r only when $a = 1$, that is $n = 2a = 2$ (verified symbolically).

This is an important difference from Paper 01. There, $\rho \propto r^{-n}$ is a family in which every n is a realisable density distribution, and choosing $n = 2$ is a choice among genuine alternatives. Here it is not: the Schrödinger equation has already forced $n = 2$, and the flux argument confirms it.

The table of Corollary 3.2 therefore shows that the algebraic condition is identical, but not that the two statements are “uniqueness theorems” in the same sense. Paper 01 selects; this paper checks. The algebra is the same; what it acts on is not.

Observation 3.6 (third edition: the 4π does not vanish here)

The corpus carries many expressions containing 4π . Across the three most recent papers the same symbol met three different ends:

Paper	Where the 4π comes from	Fate
51 (shell theorem)	the sphere's area <i>and</i> Gauss's normalisation	cancels; absent from the answer
52 (Hawking temperature)	twice the Euclidean period 2π	not a solid angle at all
This paper (probability flux)	the sphere's area alone	nothing to cancel it; it survives

The 4π of equation (8) is **the solid angle Ω_2 , a fact**, with no convention multiplying it. And unlike Paper 51, **there is no partner to annihilate it**: no normalisation on the other side, as in Gauss's law, carries the same 4π . **Whether a 4π survives or vanishes is decided by how many times it occurs in the equation.**

So the phrase in Corollary 3.2, that “the solid angle 4π appears in both”, becomes checkable only once the address is written (Paper 17). What is common is **integrating the Ω of the area element $r^2 d\Omega$ over the full solid angle**, not the agreement of a number 4π .

4. Relation to prior work, and the scope of this paper

Correction record 4.1 (first edition, unchanged): The first edition identified the modulus $|\psi|$ with an “effective displacement radius r ” and claimed to derive the Born rule from the shell area $4\pi r^2 = 4\pi |\psi|^2$. **That identification is dimensionally inconsistent and stands withdrawn.** From the normalisation condition in three dimensions, $|\psi|^2$ has dimension $[L]^{-3}$ and hence $|\psi|$ has $[L]^{-3/2}$, which cannot be identified with a length $[L]$. The salvageable part of the intuition was made rigorous as Theorem 3.1.

4.1 Prior derivations of the Born rule

- **Gleason's theorem** (Gleason, 1957): on a separable Hilbert space of dimension at least 3, every probability measure on the lattice of projections has the form $\mu(P) = \text{Tr}(\rho P)$. Given the Hilbert-space structure and non-contextuality, the Born rule is **mathematically forced**. It remains the strongest result.
- **Decision-theoretic derivations** (Deutsch, 1999; Wallace, 2007): within the Everett interpretation, from the preference axioms of a rational agent.
- **Envariance** (Zurek, 2005): from the symmetry of a system entangled with its environment.
- **Operational derivations** (Masanes, Galley & Müller, 2019): from operational axioms about measurement.

4.2 What this paper claims

This paper claims no complete derivation of the Born rule. As Gleason's theorem shows, its mathematical necessity is already strongly constrained by the Hilbert-space structure, and nothing here goes beyond that.

The claims are limited to the following:

1. **Within the family of power functionals**, unitarity fixes the usable exponent at $n = 2$ (Theorem 2.1). **But this assumes the probability density is a power of $|\psi|$, and is weaker than Gleason's theorem for that reason** (Remark 2.4). What does the fixing is **the demand for all states**, not any single one (Observation 2.3).
2. The flux-conservation condition under the Born rule is the same algebraic condition as Paper 01's $n = 2$ uniqueness theorem (Theorem 3.1). **But the two are not “uniqueness” in the same sense** (Observation 3.5).
3. **[Third edition] The two 2s come from different roots and part company in four dimensions** (Observation 3.4).
4. **[Third edition] Whether a 4π survives or cancels is decided by how many times it occurs** (Observation 3.6).

What this paper does not show: (i) that the probability density is a power of $|\psi|$ (the assumption of Theorem 2.1); (ii) why a conserved quantity should be read as *probability* (measurement theory); (iii) a derivation of the Born rule — **Theorem 3.1 is a consistency check that presupposes it** (Remark 3.3); (iv) that the agreement of the two routes is more than a fact about our dimension — **they agree only at $d = 3$** (Observation 3.4).

5. Interpretation in the universe OS — Layer 3 (non-scientific)

What follows is interpretation, not theorem, and is quarantined as Layer 3 (non-scientific). Nothing above uses any of it. A physical detector always interacts with a system on a two-dimensional boundary. As Theorem 3.1 shows, the flux through that surface couples to the probability density through the area element $r^2 d\Omega$, and the sum over the full solid angle is the conserved quantity. In the universe-OS picture this may be read as **the bandwidth allocation when data distributed in three dimensions is projected onto a two-dimensional detection surface.** [Third-edition correction: calling a notation of Paper 41 an “operator”]: The second edition wrote here of “the two-dimensional projection operator $\mathcal{P}_{2D}$ of Paper 41”. **The third edition of Paper 41 states explicitly that $\mathcal{P}_{2D}$ is not a projection operator** — it fails idempotency $\mathcal{P}^2 = \mathcal{P}$, and a δ function is not even an element of the state space, so it is **a notational convenience, neither a theorem nor a derivation.** This edition calls it “the notation $\mathcal{P}_{2D}$ ” and not an operator. **The correction exists so that a withdrawn construction is not called back through a citation.** This paper does not claim that the reading **explains** the Born rule. What Theorem 3.1 establishes is the consistency of the Born rule with three-dimensional spherical geometry, not a derivation of the former from the latter.

6. Conclusion

This paper establishes the following.

1. The only power functional $\int |\psi|^n$ conserved under unitary evolution **for every initial state** is $n = 2$ (Theorem 2.1, Corollary 2.2, verified symbolically). **What does the fixing is the demand for “every”; looking only at stationary states, all n are conserved** (Observation 2.3).
2. The flux-conservation condition $|\psi|^2 \propto r^{-2}$ for the three-dimensional spherical wave is the same algebraic condition as the $n = 2$ uniqueness theorem of Paper 01 (Theorem 3.1, Corollary 3.2).
3. [Third edition] **The two 2s come from different roots** — Theorem 2.1's is fixed by **dynamics** and is independent of dimension; Theorem 3.1's is fixed by **geometry** and becomes $n = d - 1$ in d dimensions. **They coincide only in three dimensions, where $d - 1 = 2$; in four they split into 2 and 3** (Observation 3.4).
4. [Third edition] **The alternatives Theorem 3.1 excludes do not exist as free-particle states** — $\psi = Ae^{ikr}r^{-n/2}$ solves the free Schrödinger equation only for $n = 2$ (Observation 3.5, verified symbolically). Paper 01 selects among genuine alternatives; this paper checks. **The algebra is the same; what it acts on is not.**
5. [Third edition] **The 4π does not vanish here** — in equation (8) it occurs once, from the sphere's area, with nothing to cancel it. In Paper 51 it occurred twice and vanished; in Paper 52 it was not a solid angle at all. **One symbol, three fates** (Observation 3.6).

6. [Third edition] The phrase “two-dimensional projection operator” in Section 5 was corrected — the third edition of Paper 41 states that $\mathcal{P}_{2D}$ is a notation, not an operator.
7. In the light of prior work beginning with Gleason's theorem, it is made explicit that this paper claims no complete derivation of the Born rule (Section 4).

Closing: There are two structural grounds for the exponent of the Born rule being 2. But they are not two paraphrases of one reason. **One is fixed by dynamics and the other by geometry, and they point at the same number because we live in three dimensions.** On seeing the same number, ask first for its address — that is the one thing this paper returns to the corpus.

Revision Record

Added in the third edition (August 2026)

1. **The title is unchanged; only the subtitle was replaced**, to name the content of this edition (the two roots, and where the 4π goes). The main title stands as it is.
2. **Observation 3.4 added:** Theorem 2.1's 2 is dynamics, Theorem 3.1's is geometry; the former is dimension-independent, the latter becomes $n = d - 1$. **Four dimensions is the counterexample that separates them.**
3. **Observation 3.5 added:** $\psi = Ae^{ikr}r^{-n/2}$ solves the free Schrödinger equation only for $n = 2$ (verified symbolically). The difference in the standing of “uniqueness” relative to Paper 01 is made explicit.
4. **Observation 3.6 added:** the three fates of 4π (Paper 51 cancels / Paper 52 not a solid angle / here it survives).
5. **Observation 2.3 added:** in a stationary state every n is conserved; the force of Theorem 2.1 lies in “for every initial state”.
6. **“The two-dimensional projection operator $\mathcal{P}_{2D}$ ” in Section 5 was corrected to “the notation $\mathcal{P}_{2D}$ ”,** the third edition of Paper 41 having stated that it is not an operator.
7. **Section 5 was explicitly quarantined as Layer 3.**
8. **The counterexample of Theorem 2.1 was re-verified symbolically:** for $n = 1, 3/2, 2, 5/2, 3, 4$ the powers of σ are $3/2, 3/4, 0, -3/4, -3/2, -3$.
9. **DOIs were added to the self-citations and the format unified.**

Corrected in the second edition (unchanged)

1. The first edition's identification $r = |\psi|$ is dimensionally inconsistent and stands withdrawn (Correction record 4.1).
2. What the first edition called “the probability flux”, $\int_{S^2} |\psi|^2 r^2 d\Omega$, is not a flux (its dimension is $[L^{-1}]$). The form using the probability current $\mathbf{j}$ was substituted (proof of Theorem 3.1).

3. It was made explicit that Theorem 2.1 is a uniqueness statement inside the family of power functionals, and is weaker than Gleason's theorem (Remark 2.4).

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