The Upper and Lower Bounds of a Composite Are the Arithmetic and Harmonic Means Themselves

— the upper bound is the arithmetic mean and the lower the harmonic, with the geometric mean always between — the spread depends only on the ratio of the materials, staying at **16.932376** even when both stiffnesses are multiplied by ten — [Paper 540]

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— *From thinking to syncing.* —

Abstract

A composite's stiffness has an upper and a lower bound, and their ratio has a closed form.What this paper shows is that those two bounds are the arithmetic and harmonic means themselves, so the closed form has an address within the family of means.No new theorem or law is claimed. First, the numbers of Paper 342 return——carbon fibre at 230 GPa and epoxy at 3.5 GPa in equal parts give 116.7500 GPa in parallel and 6.8951 GPa in series (Section 2). **Second, and this is the core.That closed form is the arithmetic mean divided by the harmonic mean——$\frac{1}{2} + \frac{1}{4}(r + 1/r) = 16.932376$ agrees with 116.7500/6.8951 to 12 digits** (Section 2). **Third, the geometric mean always lies between——at a volume fraction of 0.5 it is 28.3725 GPa, inside 6.8951 and 116.7500** (Section 3). **Fourth, the spread depends only on the ratio of the materials——$(230, 3.5)$, $(460, 7.0)$ and $(23, 0.35)$ all give a maximum spread of 16.932376** (Section 4). **Fifth, changing the ratio moves it——a material ratio of 10 gives a maximum spread of 3.025000** (Section 4). **Sixth, the separator is which mean is taken——equal strain corresponds to the arithmetic mean and equal stress to the harmonic** (Section 5). **Keywords**: composite material, Voigt-Reuss bounds, arithmetic mean, harmonic mean, elastic modulus.

Scope of this paper (scope note): **No new theorem or law is claimed——the Voigt and Reuss bounds and the inequalities between means are standard.No tighter bounds are proposed——the Hashin--Shtrikman bounds are not entered; only Voigt and Reuss are examined.No experimental values are treated——where a real composite falls is not the subject.Anisotropy is not treated——only uniaxial stiffness is considered.No material is recommended——which volume fraction to build at is not stated.It is not said that being a mean makes it correct——that these are bounds follows from mechanics, not from any property of means.Relation to earlier papers**: **Paper 342 showed that mixing the same two materials in the same proportion opens a stiffness ratio of $\mathbf{16.932376}$ and gave a closed form——this paper shows that the closed form is a ratio of arithmetic to harmonic mean, a naming that paper did not make.Paper 194 showed that the four means are one family with a single equality fence——the bounds here are the two ends of that family, and the geometric mean lies inside by the family's own inequality.Paper 300 showed that sharing a root is decidable——the verdict here is reached under the criterion of a correspondence that preserves identity.**What is added is **rebuilding the numbers of Paper 342 independently**, **confirming to 12 digits that its closed form is arithmetic over harmonic**, **confirming that the geometric mean lies between at every fraction**, **showing the spread unchanged under a tenfold scaling**, and **placing the separator on which mean is taken**.

1. Notation and the Question

Setting: **mixing two materials at volume fractions f and $1-f$, equal strain (parallel) gives a stiffness $fE_1+(1-f)E_2$ and equal stress (series) gives $(f/E_1+(1-f)/E_2)^{-1}$. The first is the Voigt upper bound, the second the Reuss lower bound.**

Observation 1.1

the formulas already have the shape of means.Parallel is a weighted arithmetic mean, series a weighted harmonic mean——two expressions from mechanics landing at the two ends of an arithmetic family.

Observation 1.2

Paper 342 gave the closed form of their ratio.In the shape $\frac{1}{2}+\frac{1}{4}(r+1/r)$——without using the word "harmonic mean". The thing had no name.

Question: **where in the family of means does that closed form live?**

2. Rebuilding the Numbers of Paper 342

Computation 2.1: carbon fibre at $E_1 = 230$ GPa and epoxy at $E_2 = 3.5$ GPa, at a volume fraction of 0.5.

quantity	computed here	Paper 342
parallel (Voigt)	**116.7500** GPa	116.7500
series (Reuss)	**6.8951** GPa	6.8951
closed form of the ratio	**16.932376**	16.932376

> **Observation 2.2**
>
> all three return.The earlier numbers were rebuilt independently and agreed——disagreement would mean either the material values or the formula was wrong. A check that can fail.

Computation 2.3: comparing parallel divided by series against the closed form,

$$\frac{\frac{1}{2}(E_1 + E_2)}{\left(\frac{1}{2}(1/E_1 + 1/E_2)\right)^{-1}} = \frac{(E_1 + E_2)^2}{4E_1E_2} = \frac{1}{2} + \frac{1}{4}\left(r + \frac{1}{r}\right)$$

both giving **16.932376**, agreeing to 12 digits.

> **Observation 2.4**
>
> this is the core.The closed form of Paper 342 is the arithmetic mean over the harmonic mean——they carry into each other by algebra, so this is not agreement of numbers.

> **Observation 2.5**
>
> so it acquires a name.“The ratio of the bounds” is “the ratio of arithmetic to harmonic mean”——an address inside the family of Paper 194.

3. The Geometric Mean Lies Between

Computation 3.1: listing the three means across volume fractions.

f	Voigt	Reuss	ratio	geometric
0.1	26.1500	3.8823	6.7357	5.3190
0.3	71.4500	4.9676	14.3832	12.2847
0.5	**116.7500**	**6.8951**	**16.9324**	**28.3725**
0.7	162.0500	11.2666	14.3832	65.5285
0.9	207.3500	30.7839	6.7357	151.3428

Observation 3.2

the geometric mean is inside at every fraction.At $f = 0.5$, $\mathbf{6.8951 < 28.3725 < 116.7500}$——this comes not from mechanics but from the inequality within the family of means.

Observation 3.3

the ratio is symmetric in f and $1 - f$.$f = 0.1$ and $f = 0.9$ both give 6.7357, $f = 0.3$ and $f = 0.7$ both give 14.3832——exactly the symmetry Paper 342 identified.

Observation 3.4

the maximum sits at the centre.$\mathbf{16.9324}$ at $f = 0.5$——at the ends only one material remains and the ratio approaches one. The spread is largest with equal parts.

Observation 3.5

the geometric mean is not a bound.It lies between, without guaranteeing that measurements land there——mechanics guarantees only the two ends; the interior is mathematics, not materials.

4. What Fixes the Spread

Computation 4.1: **scaling both stiffnesses and re-measuring the maximum spread.**

materials (GPa)	ratio	maximum spread
$(230, 3.5)$	65.714	**16.932376**
$(460, 7.0)$	65.714	**16.932376**
$(23, 0.35)$	65.714	**16.932376**
a ratio of 10 instead	10	**3.025000**

> **Observation 4.2**
>
> **multiplying both stiffnesses by ten leaves the spread untouched.Still $\mathbf{16.932376}$——the spread is dimensionless and carries only the ratio of the materials.**

> **Observation 4.3**
>
> **changing the ratio does move it.A ratio of 10 gives $\mathbf{3.025000}$——confirming the check is not vacuous: a quantity nothing moves is not being tested.**

> **Observation 4.4**
>
> **the formula shows it too.Only r enters $\frac{1}{2} + \frac{1}{4}(r + 1/r)$——neither E_1 nor E_2 appears alone.**

5. The Separator

> **Observation 5.1**
>
> **the separator is which mean is taken.Equal strain gives the arithmetic mean, equal stress the harmonic——the choice of boundary condition corresponds to the choice of mean.**

Observation 5.2

the fence of Paper 194 applies here.There the four means form one family with a single equality fence——that fence is the case of two materials of equal stiffness, where the bounds coincide and the ratio is one.

Observation 5.3

the second criterion of Paper 300 is met.Algebra carries the mechanical expressions into the expressions for means——not agreement of numbers.

Observation 5.4

the direction needs care.That these are bounds follows from mechanics——belonging to a family of means is something learned afterwards. They are not bounds because they are means.

Observation 5.5

what is not said.Tighter bounds, experimental values and anisotropy are not treated——nor is any volume fraction recommended.

6. Roll Call and Closing

1. **parallel 116.7500 GPa** (Computation 2.1): **matching Paper 342.**
2. **series 6.8951 GPa** (Computation 2.1): **matching Paper 342.**
3. **closed form 16.932376** (Computation 2.1): **matching Paper 342.**
4. **12 digits with arithmetic over harmonic** (Computation 2.3): **the core.**
5. **geometric mean 28.3725** (Computation 3.1): **always between.**
6. **symmetric in f and $1 - f$** (Observation 3.3): **0.1 matches 0.9.**
7. **maximum at $f = 0.5$** (Observation 3.4): **ends approach one.**
8. **tenfold scaling leaves 16.932376** (Computation 4.1): **dimensionless.**
9. **a ratio of 10 gives 3.025000** (Observation 4.3): **not vacuous.**

10. **the separator is the mean** (Observation 5.1): **chosen by boundary condition.**

11. **not bounds because means** (Observation 5.4): **the direction is from mechanics.**

Closing: **A composite's stiffness has an upper and a lower bound, and their ratio has a closed form.The earlier numbers return——carbon fibre at 230 GPa and epoxy at 3.5 GPa in equal parts give 116.7500 GPa in parallel, 6.8951 GPa in series and a ratio of 16.932376, matching the three numbers of Paper 342. Rebuilding them independently checks both the material values and the formula, and it is a check that can fail.That closed form is the arithmetic mean over the harmonic mean——$(E_1 + E_2)^2/(4E_1E_2)$ equals $\frac{1}{2} + \frac{1}{4}(r + 1/r)$, both giving 16.932376 to 12 digits. They carry into each other by algebra, so the second criterion of Paper 300 is met and this is not agreement of numbers. The ratio of the bounds now has a name, which Paper 342 did not give it.The geometric mean always lies between——at $f = 0.5$, $6.8951 < 28.3725 < 116.7500$, which follows from the inequality within the family of Paper 194 rather than from mechanics. The ratio is symmetric in f and $1 - f$, with 0.1 and 0.9 both at 6.7357 and 0.3 and 0.7 both at 14.3832. The maximum sits at the centre, since at the ends one material remains and the ratio approaches one. The geometric mean is not itself a bound: it lies between without guaranteeing that measurements land there.What fixes the spread is the ratio of the materials alone——$(230, 3.5)$, $(460, 7.0)$ and $(23, 0.35)$ all give 16.932376, while a ratio of 10 gives 3.025000. A quantity nothing moves is not being tested, so it was confirmed that something does move it. The formula agrees: only r enters, and neither stiffness appears alone.The separator is which mean is taken——equal strain gives the arithmetic mean and equal stress the harmonic, so the boundary condition chooses the mean. The equality fence of Paper 194 is the case of equal stiffnesses, where the bounds coincide and the ratio is one. The direction needs care: these are bounds because of mechanics, and their membership in a family of means is learned afterwards.To be honest——no tighter bounds were proposed; the Hashin--Shtrikman bounds were not entered and only Voigt and Reuss examined. No experimental values were treated and where a real composite falls is not the subject. Anisotropy was not considered, only uniaxial stiffness, and no volume fraction is recommended. It is not claimed that being a mean makes anything correct. What can be said is that two bounds from mechanics are the arithmetic and harmonic means themselves, and that their ratio depends only on the ratio of the materials, and no further.**

References

1. Hardy, G. H., Littlewood, J. E., & Polya, G. (1934). *Inequalities*. Cambridge University Press.

2. Hill, R. (1952). *The Elastic Behaviour of a Crystalline Aggregate*. Proceedings of the Physical Society A.

3. Reuss, A. (1929). *Berechnung der Fliessgrenze von Mischkristallen*. Zeitschrift fur Angewandte Mathematik und Mechanik.

4. Voigt, W. (1889). *Uber die Beziehung zwischen den beiden Elasticitatsconstanten isotroper Korper*. Annalen der Physik.

5. Yamagishi, Y. (2026). *The Four Means Are One Family, and There Is Only One Equality Fence* [Paper 194]. Zenodo. 10.5281/zenodo.22106809

6. Yamagishi, Y. (2026). *The Three Hundredth Census: Whether Two Things Share a Root Is Decidable* [Paper 300]. Zenodo. 10.5281/zenodo.22171914

7. Yamagishi, Y. (2026). *Mix the Same Two Materials in the Same Proportion and the Stiffness Differs by* **16.93** [Paper 342]. Zenodo. 10.5281/zenodo.22682160

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