Whether a Top Stands or Falls Is Decided Not by How Fast It Spins but by One Threshold

— For a top made of a disc of mass **50** g and radius **2.5** cm, the threshold is **638.3542** revolutions per minute

— the approximate formula for precession keeps giving answers below the threshold — [Paper 586]

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— *From thinking to syncing.* —

Abstract

A top that spins fast is said not to fall.What this paper shows is that whether a top can stand upright is decided not by how fast it spins but by one threshold, and that below the threshold a top released from however small a tilt falls to a definite angle.No new theorem or law is claimed. First, there is a threshold——it is $\omega_c = 2\sqrt{I_1 Mgl}/I_3$, which for a top made of a disc of mass 50 g and radius 2.5 cm is 66.848293 rad/s, or 638.3542 revolutions per minute (Section 2). **Second, above the threshold the top only wobbles a little——released gently near the vertical, its greatest tilt is $q/\sqrt{q^2-1}$ times the initial tilt (q being the spin relative to the threshold): 1.809068 times at $q = 1.2$ and 7.123991 times at $q = 1.01$** (Section 3). **Third, and this is the core.Below the threshold, however small the initial tilt, the greatest tilt does not shrink below $2\cos^{-1} q$——16.219229 degrees at $q = 0.99$ and 51.683866 degrees at $q = 0.9$; solving the equations of motion directly agrees to 2.5×10^{-9} degrees** (Section 3). **Fourth, a little below the threshold means not "falls more easily" but "falls"——the rim of this top's disc touches the floor at 38.6598 degrees, so at 602.3691 revolutions per minute, 5.6372 per cent below the threshold, the rim reaches the floor even when the top is released from the vertical** (Section 3). **Fifth, the approximate formula for precession has no threshold——at a tilt of 20 degrees and 3000 revolutions per minute, the exact precession is 1.010869 times the approximation $Mgl/(I_3\omega)$, a good approximation; at the threshold the exact precession is exactly 2.000000 times the approximation; and at 300 revolutions per minute, below the threshold, the formula still answers 10.000000 times the precession at 3000, when no steady precession exists** (Section 4). **Sixth, the separator is the sign of the discriminant $I_3^2\omega^2 - 4I_1 Mgl\cos\theta$——the same discriminant decides whether the top can stand upright ($\theta = 0$) and whether it can precess steadily at a tilt** (Section 5). **Keywords**: spinning top, heavy symmetric top, sleeping top, stability threshold, precession.

Scope of this paper (scope note): **No new theorem or law is claimed——the motion of the heavy symmetric top (Lagrange's top), the stability condition of the sleeping top, and the approximate formula for precession are all standard.Friction is not included——the loss of spin to friction at the point and to air, and the rising of a tilted top by friction, are not treated.The point does not move——the tip is taken to stay at one point on the floor.The dimensions are representative——a thin disc of mass 50 g and radius 2.5 cm, held 2 cm above the point on a weightless axle.The floor enters only through the tilt at which the rim touches it; what happens after is not treated.No top is designed.Relation to earlier papers**: **Paper 359 showed that the gyroscope is two orders too small to explain why a bicycle does not fall——the top is a system that stands by the gyroscopic effect alone, but only above the threshold.Paper 300 showed that sharing a root is decidable——"can it stand upright" and "can it precess steadily at a tilt" come from the same discriminant and share a root.**What is added is **confirming by two routes, the quadratic from the conserved quantities and energy and the equations of motion, that above the threshold the wobble stays at $q/\sqrt{q^2-1}$ times the initial tilt and below it the top falls to $2\cos^{-1} q$, finding the spin at which the rim reaches the floor, showing that the approximate formula for precession has no threshold**, and **placing the separator on the sign of the discriminant**.

1. Notation and the Question

Setting: **a symmetric top spins about a point on the floor.Let its mass be M, the height of its centre of mass above the point l, its moment of inertia about its axis I_3, about a horizontal axis through the point I_1, its spin ω, and its tilt from the vertical θ.The angular momentum about the axis, the angular momentum about the vertical, and the energy are conserved.**

> **Observation 1.1**
>
> **a top that spins fast is said not to fall.It sounds like a matter of degree, the faster the steadier——whether it is a matter of degree or of a boundary is another question.**

Question: **is whether a top can stand upright decided by how fast it spins, or by one threshold?**

2. The Threshold

Computation 2.1: **a thin disc of mass 0.050 kg and radius 0.025 m held 0.020 m above the point.With $I_3 = 1.5625 \times 10^{-5}$ kg m^2, $I_1 = 2.78125 \times 10^{-5}$ kg m^2 and $Mgl = 0.00980665$ N m, the threshold**

$$\omega_c = \frac{2\sqrt{I_1 Mgl}}{I_3}$$

is **66.848293** rad/s, or **638.3542** revolutions per minute.

> **Observation 2.2**
>
> the threshold is set by the balance between the strength with which gravity tips the top, Mgl, and the angular momentum of its spin, $I_3\omega$.The boundary is whether $I_3\omega$ exceeds $2\sqrt{I_1 Mgl}$——the next section shows that above and below it the top behaves differently not in degree but in kind.

3. Releasing Near the Vertical

Computation 3.1: the top was tilted by θ_0 from the vertical and released gently, with neither precession nor wobble.From the conserved quantities and energy, the greatest tilt is a root of a quadratic.With $q = \omega/\omega_c$,

q	rev/min	greatest at $\theta_0 = 0.01$ rad	limit $\theta_0 \to 0$
1.20	766.0250	1.036433 degrees	0 degrees
1.01	644.7377	3.966115 degrees	0 degrees
0.99	631.9706	16.689836 degrees	**16.219229** degrees
0.95	606.4365	36.434312 degrees	**36.389745** degrees
0.90	574.5188	51.699421 degrees	**51.683866** degrees
0.80	510.6833	73.745100 degrees	**73.739795** degrees
0.50	319.1771	120.001103 degrees	**120.000000** degrees

> **Observation 3.2**
>
> above the threshold, a small initial tilt gives a small wobble.In the limit of small tilts the greatest tilt is $q/\sqrt{q^2-1}$ times the initial one: **1.809068** times at $q = 1.2$, **3.279649** at $q = 1.05$ and **7.123991** at $q = 1.01$ (at an initial tilt of 10^{-3} rad the quadratic gives 1.809067, 3.279607 and 7.121760)——it grows as the threshold nears, but it is finite at every spin, and a smaller initial tilt makes the wobble as small as one likes.

Observation 3.3

this is the core.Below the threshold, however small the initial tilt, the greatest tilt does not shrink below $2\cos^{-1} q$——at $q = 0.99$, only one per cent below the threshold, the top falls to 16.219229 degrees from however small a tilt. As the spin crosses the threshold, the wobble changes in kind, from “so many times the initial tilt” to “a definite angle”.

Computation 3.4: as a route separate from the quadratic, the equations of motion were solved numerically (Runge--Kutta, initial tilt 0.01 rad).The greatest tilt is 1.036433 degrees at $q = 1.2$ and 51.699421 degrees at $q = 0.9$, differing from the quadratic by at most 2.5×10^{-9} degrees.

Observation 3.5

for this top, the rim of the disc touches the floor at a tilt of $\tan^{-1}(l/a) = 38.6598$ degrees.Released from the vertical, the rim reaches the floor when $2\cos^{-1} q$ exceeds this, that is, when q is 0.943628 or less——at 602.3691 revolutions per minute, 5.6372 per cent below the threshold, the top puts its rim on the floor. A little below the threshold means not “a little less steady” but “falls”.

4. The Approximate Formula for Precession

Computation 4.1: the rate of steady precession at a tilt of 20 degrees, from the slow root of the exact quadratic and from the fast-top approximation $Mgl/(I_3\omega)$.The threshold at 20 degrees is $\omega_c\sqrt{\cos 20^\circ}$, or 618.8061 revolutions per minute.

rev/min	approximate (rad/s)	exact (rad/s)	exact/approximate
3000	1.997794	2.019509	1.010869
1000	5.993383	6.713230	1.120107
700	8.561976	11.669027	1.362890
650	9.220589	14.119609	1.531313
600	9.988972	no root	none
300	19.977943	no root	none

Observation 4.2

for a fast top the approximation is good.At 3000 revolutions per minute the exact precession is 1.010869 times the approximation, with a period of 3.1112 seconds——the approximation says that precession quickens as spin falls, and in this range that is right.

Observation 4.3

the approximation has no threshold.At the threshold the two roots merge and the exact precession is exactly 2.000000 times the approximation. Below it, at 300 revolutions per minute, the approximation still answers 10.000000 times the precession at 3000——while the exact equation has no root for steady precession at all. The approximation goes on giving plausible numbers at spins at which the top falls.

5. The Separator

Observation 5.1

the separator is the sign of the discriminant $I_3^2\omega^2 - 4I_1 Mgl\cos\theta$.If positive, the top can precess steadily at tilt θ; if negative, it cannot. Setting $\theta = 0$ returns the threshold ω_c——standing upright and precessing at a tilt are decided by the same discriminant.

Observation 5.2

beside Paper 359.It showed that the gyroscope is two orders too small to explain why a bicycle does not fall ——the top is a system that stands by the gyroscopic effect alone, but only above the threshold; below it, the top falls gyroscope and all.

Observation 5.3

one root, under Paper 300.“Can it stand upright” and “can it precess steadily at a tilt” come from the same discriminant——what separates them is only the value of $\cos\theta$.

> **Observation 5.4**
>
> **what is not said.Without friction, neither the slow loss of spin that carries a real top below the threshold nor the rising of a tilted top is treated, and no top is designed——what is shown is that standing and falling are separated by a threshold, across which the behaviour changes in kind.**

6. Roll Call and Closing

1. **the threshold is 638.3542 revolutions per minute** (Computation 2.1): $\omega_c = 2\sqrt{I_1 Mgl}/I_3$.
2. **above it, $q/\sqrt{q^2-1}$ times the initial tilt** (Observation 3.2): **only 7.123991 times even at $q = 1.01$.**
3. **below it, a fall to $2\cos^{-1} q$** (Observation 3.3): **the core.**
4. **one per cent slow gives 16.219229 degrees** (Computation 3.1): **whatever the initial tilt.**
5. **equations of motion and quadratic differ by 2.5×10^{-9} degrees** (Computation 3.4): **two routes.**
6. **the rim touches the floor at 38.6598 degrees** (Observation 3.5): **5.6372 per cent below the threshold, the top falls.**
7. **at 3000 revolutions per minute, exact is 1.010869 times the approximation** (Computation 4.1): **a good approximation.**
8. **at the threshold, exact is 2.000000 times the approximation** (Observation 4.3): **the two roots merge.**
9. **the approximation answers below the threshold** (Observation 4.3): **at spins with no steady precession.**
10. **the separator is the sign of the discriminant** (Observation 5.1): **standing and precessing share a root.**

Closing: **A top that spins fast is said not to fall.That is a matter not of degree but of a threshold——for a top made of a disc of mass 50 g and radius 2.5 cm, the threshold is $\omega_c = 2\sqrt{I_1 Mgl}/I_3$, or 638.3542 revolutions per minute. Above it, a top released near the vertical wobbles only $q/\sqrt{q^2-1}$ times its initial tilt, no more than 7.123991 times even at $q = 1.01$, one per cent fast. Below it, from however small a tilt, it falls to $2\cos^{-1} q$, reaching 16.219229 degrees at only one per cent slow; since this top's rim touches the floor at 38.6598 degrees, it falls at 602.3691 revolutions per minute, 5.6372 per cent below the threshold.The approximate formula for precession, $Mgl/(I_3\omega)$, has no such threshold——it is good at 3000 revolutions per minute, falls to exactly half the exact value at the threshold, and below it goes on answering plausible numbers at spins at which no steady precession exists.The separator is the sign of the discriminant $I_3^2\omega^2 - 4I_1 Mgl\cos\theta$——the same discriminant decides whether the top can stand upright and whether it can precess at a tilt; under Paper 300 the two share a root.Placed among the earlier papers——Paper 359 showed that the gyroscope is two orders too small to explain why a bicycle does not fall; the top is a system that stands by the gyroscopic effect alone, but only above the threshold.To be honest——without friction, neither the slow loss of spin that carries a real top below the threshold nor the rising of a tilted top is**

treated. The dimensions are representative, the point does not move, and what happens after the rim touches the floor is not treated.

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