

The Volume of the Unit Ball Is Greatest in Five Dimensions

— 2nd ed.: That Five Is a Consequence of Having Set the Radius to One (the Yamagishi Patch and the Velocity Ball) — [Paper 59]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

ABSTRACT

That the volume $V_n = \pi^{n/2}/\Gamma(n/2 + 1)$ of the unit ball in n dimensions is greatest at $n = 5$ among the integers (and at $n^* \approx 5.2569$ as a continuous variable) is **classically known**. The first edition placed this within the corpus, noting that reading c as a radius makes c^n the scaling of an n -dimensional ball's volume, and that “**the dimension of greatest volume, five**” corresponds to the intuition of “**greatest capacity, a server**” — while rightly warning that $d = 5$ must not be confused with the $d = 3$ of stability. The second edition pushes that warning further than the first edition anticipated. First, the answer $n = 5$ is a consequence of having set the radius to one. Comparing balls of radius λ instead, the maximising condition becomes $\psi(n/2 + 1) = \ln(\pi\lambda^2)$ and the peak moves: 5.26 at $\lambda = 1$, 24.1 at $\lambda = 2$, 55.5 at $\lambda = 3$, 627 at $\lambda = 10$, and 0.48 at $\lambda = 1/2$ (all verified numerically here). Measure the same ball with a different ruler and the peak goes wherever you like. So $d = 5$ is a convention, not a fact (Paper 95). Second, V_4 and V_5 cannot be compared at all. V_n is a dimensionless coefficient; the volume is $V_n c^n$, of dimension L^n . The comparison sets a number attached to L^4 beside one attached to L^5 — the category error named by Paper 28 (an exponent is not a dimension) and Paper 45 (an area is not a solid angle) (Observation 2.3). Third, this sharpens the first edition's Section 4. That section said, correctly, that changing the criterion changes the special dimension. The second edition refines the classification: **some criteria survive a change of ruler and some do not**. That all bound orbits close (Bertrand), that a bound ground state exists, and that Huygens' principle holds are **independent of the ruler**. That the unit ball's volume is greatest is not. $d = 3$ is a fact about the world; $d = 5$ is a fact about the number one (Observation 4.2). Fourth, reading c as a radius can be put more strongly than the first edition thought. It weakened this to “a choice of representation, on a par with $G = 4\pi$ ”. No weakening is needed: in special relativity $|v| < c$, so the set of attainable velocities really is a ball of radius c , whose volume $V_n c^n$ is dimensionally consistent as $(L/T)^n$. This is the corpus's own velocity sphere (Papers 20, 63). It is not a patch but a real object (Observation 3.2). Accordingly, the first edition's Corollary 1 (“the five-dimensional server”) is withdrawn.

Scope of this paper: NO NEW PHYSICAL LAW AND NO NEW MATHEMATICAL THEOREM IS CLAIMED. That V_n peaks at $n = 5$ is classically known (Smith & Vamanamurthy 1989). What the second edition adds is only a judgement about **the range within which that known fact means anything**.

1. Notation and position

Position: This paper claims no new physical law and no new mathematical theorem. Its contribution is to place a known fact within the corpus and to **separate what is a re-reading from what is merely a convention**.

Added in the second edition: The first edition rightly warned that $d = 5$ must not be confused with $d = 3$. **The second edition turns that warning on $d = 5$ itself.**

2. The known fact: the unit ball's volume peaks in five dimensions

Theorem 2.1 (dimensional dependence of the unit ball's volume, classically known)

The volume of the unit ball in n dimensions is

$$V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \tag{1}$$

greatest among the integers at $n = 5$, with $V_5 = 8\pi^2/15 \approx 5.2638$; as a continuous variable, $\psi(n/2 + 1) = \ln \pi$ gives $n^* \approx 5.2569$ (verified numerically here).

n	V_n	Numeric
2	π	3.1416
3	$4\pi/3$	4.1888
4	$\pi^2/2$	4.9348
5	$8\pi^2/15$	5.2638 (greatest)
6	$\pi^3/6$	5.1677
7	$16\pi^3/105$	4.7248

As $n \rightarrow \infty$, $V_n \rightarrow 0$ ($V_{20} = 2.58 \times 10^{-2}$, $V_{50} = 1.73 \times 10^{-13}$, verified here).

2.1 [Second edition] What exactly is being compared?

Observation 2.2 (V_n is a coefficient, not a volume)

The V_n of (1) is a **dimensionless coefficient**. The actual volume of a ball of radius r is $V_n r^n$, of dimension L^n .

Observation 2.3 (so comparing V_4 with V_5 is a category error)

“ $V_5 > V_4$ ” sets a **number attached to L^5 beside one attached to L^4** . This is the error the corpus has named repeatedly: Paper 28 judged that “**3/2 is a scaling exponent, not a dimension**”, and Paper 45 that “**an area is not a solid angle**”. **Quantities of different dimension cannot be compared as they stand.**

The comparison looks possible only because the radius was set to one — and, as the next subsection shows, **that one is a choice of ruler**.

2.2 [Second edition] The peak moves with the ruler

Observation 2.4 ($n = 5$ is a consequence of having set the radius to one)

Comparing balls of radius λ , what is to be maximised is $V_n \lambda^n$. Differentiating $\ln(V_n \lambda^n)$ with respect to n and setting it to zero gives

$$\psi\left(\frac{n}{2} + 1\right) = \ln(\pi \lambda^2) \quad (2)$$

At $\lambda = 1$ the right-hand side is $\ln \pi$ and $n^* \approx 5.2569$; but changing λ moves the peak (all verified numerically here):

Radius unit	$\ln(\pi\lambda^2)$	Peak n^*
$\lambda = 1/2$	-0.2416	0.48
$\lambda = 1$ (1st ed.)	1.1447	5.26
$\lambda = 2$	2.5310	24.1
$\lambda = 3$	3.3420	55.5
$\lambda = 10$	5.7499	627

Measure the same ball with a different ruler and the peak goes wherever you like. Comparing balls of radius ten, the “special dimension” is **627**.

Corollary 2.5 ($d = 5$ is a convention)

The statement “the unit ball's volume is greatest in five dimensions” therefore **says nothing about five; it says something about one**. In the vocabulary of Paper 95, it is **a convention, not a fact**.

Theorem 2.1 itself is correct. What is corrected is having read its **5** as **a fact about dimension**.

3. Placing it structurally: reading c as a radius

Proposition 3.1 (c^n and the n -dimensional volume)

Reading c as a radius,

$$V_n(c) = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} c^n$$

so that c^n is the scaling of the volume of an n -dimensional ball of radius c . The passage from surface to volume, $\int 4\pi c^2 dc = (4/3)\pi c^3$, is the shell integration (co-area) of Paper 01: **integration raises the dimension by one**.

3.1 [Second edition] This reading is stronger than the first edition thought

Observation 3.2 (the velocity ball is real)

The first edition weakened “taking c as a radius” to “a choice of representation on a par with $G = 4\pi$ ”, adding that it is not a claim that c is a length. **No weakening was needed.**

In special relativity attainable velocities satisfy $|v| < c$. **That is, velocity space really contains a ball of radius c .** Its volume is $V_n c^n$, of dimension $(L/T)^n$, **which is consistent** — c need not be reinterpreted as a length at all. **It is a radius in velocity space.**

This object is already in the corpus: Paper 20 (the volume of the velocity sphere) and Paper 63 (the curvature duality between velocity space and spacetime, the “ c as radius” cluster). **It is not a patch but a real geometric object.**

Corollary 3.3 (withdrawing the first edition's Corollary 1, “the five-dimensional server”)

Even so, **the conclusion “the dimension of greatest capacity is five” does not stand**, for two reasons.

1. **The comparison itself depends on the ruler** (Observation 2.4). Holding the radius at c and varying the dimension compares a quantity of $(L/T)^4$ with one of $(L/T)^5$, and which is larger depends on the units in which c is measured. **Five appears only in a system where $c = 1$.**
2. **The dimension of velocity space equals that of position space, which is three.** A five-dimensional velocity space does not exist in our world.

The correspondence with a “five-dimensional server” is therefore withdrawn. The first edition rightly limited it to “a suggestive resonance, not a proof”; **the second edition goes further and says that the resonance itself was a product of the ruler.**

4. The distribution of “special dimensions”

Proposition 4.1 (special dimensions are distributed by criterion, unchanged)

What counts as special depends on what is being optimised or required:

- **Stability and cleanliness** (orbital stability, atomic stability, wave propagation) $\Rightarrow$ spatial $d = 3$ (Paper 56).

- **Topological normalisation** (the loop measure) $\Rightarrow$ spacetime $d = 4$ (Papers 56, 57). **But by the second edition of Paper 57 this contains a Fourier convention.**
- **Greatest unit-ball volume** $\Rightarrow d = 5$ (Theorem 2.1). **But by Observation 2.4 this depends on the choice of radius unit.**

There is no single magic dimension.

4.1 [Second edition] Refining the classification

Observation 4.2 (some criteria survive a change of ruler and some do not)

The first edition said that changing the criterion changes the special dimension. **Correct.** What the second edition adds is that **the criteria themselves are of two kinds.**

Criterion	Dimension selected	Change the ruler
All bound orbits close (Bertrand)	$d = 3$	no change
An attractive potential has a bound ground state	$d \leq 3$	no change
Huygens' principle holds	odd dimensions	no change
The unit ball's volume is greatest	$d = 5$	it moves (Obs. 2.4)

The first three hold whether length is measured in metres or in feet. Whether an orbit closes is not a question about rulers. **Only the last changes its answer when the ruler changes.**

Corollary 4.3 ($d = 3$ and $d = 5$ do not have equal standing)

The first edition was right that the two must not be confused, **but the reason is not only that the criteria differ.** $d = 3$ is a fact about the world; $d = 5$ is a fact about the number one. The two **may be listed in one table but cannot be counted with the same weight.**

5. Conclusion

1. $V_n = \pi^{n/2}/\Gamma(n/2 + 1)$ is greatest at $n = 5$ among the integers ($n^* \approx 5.2569$ continuously; verified here). This is classically known (Theorem 2.1).

2. [2nd ed.] Comparing V_4 with V_5 sets a number attached to L^4 beside one attached to L^5 — a comparison of quantities of different dimension, the category error named by Papers 28 and 45 (Observations 2.2, 2.3).
3. [2nd ed.] The answer $n = 5$ is a consequence of having set the radius to one. For radius λ the peak solves $\psi(n/2 + 1) = \ln(\pi\lambda^2)$, giving **0.48, 5.26, 24.1, 55.5 and 627** at $\lambda = 1/2, 1, 2, 3, 10$ (verified here). **$d = 5$ is a convention, not a fact** (Observation 2.4, Corollary 2.5).
4. [2nd ed.] Reading c as a radius can be put more strongly than the first edition thought: by $|v| < c$, velocity space really contains a ball of radius c , and $V_n c^n$ is dimensionally consistent as $(L/T)^n$. **It is not a patch** (Observation 3.2).
5. [2nd ed.] The first edition's Corollary 1 (“the five-dimensional server”) is withdrawn, because the comparison depends on the ruler and because velocity space has the dimension of position space, which is three (Corollary 3.3).
6. [2nd ed.] The classification of special dimensions is refined: orbital stability, atomic stability and wave propagation **do not move** when the ruler changes; the unit ball's peak **does**. **$d = 3$ is a fact about the world; $d = 5$ is a fact about the number one** (Observation 4.2, Corollary 4.3).

Closing: The first edition warned that $d = 5$ must not be confused with $d = 3$. **What the second edition did was turn that warning on $d = 5$ itself.** Turned on itself, $d = 5$ proved not even to be a fact about dimension — **it was a fact about our own decision to set the radius to one. On seeing a number that looks special, ask first what was set equal to one.**

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