Meeting the Condition That the Spheres Fill Space Exactly Does Not Mean the Code Exists

— at length 90 the sphere count also fits, $1 + 90 + 4005 = 4096 = 2^{12}$, but counting with two positions fixed gives $88/3 = 29.333333$, not an integer — counted globally it comes out whole, so the number condition alone cannot tell — [Paper 597]

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— From thinking to syncing. —

Abstract

Among error-correcting codes there are “perfect codes”, in which every word lies near exactly one codeword. Their number condition is that the points of the spheres around the codewords add up to exactly the number of words.What this paper shows is that this number condition is only necessary for a perfect code to exist: at length 90 the condition holds but there is no code, and this is seen not by a global count but only by a local count with two positions fixed.No new theorem or law is claimed. First, the Hamming code is perfect——at length 7 the radius-one spheres (8 points) around its 16 codewords cover the 128 words exactly once each: every one of the 128 words has exactly 1 codeword within distance 1 (Section 2). **Second, the Golay code is perfect too——at length 23 it has 4096 codewords of minimum weight 7; radius-three spheres have 2048 points and fill $4096 \times 2048 = 2^{23}$ exactly** (Section 3). **Third, few lengths meet the number condition——searching lengths up to 20000 and up to 8 errors corrected, apart from Hamming codes and trivial repetition codes only length 23 (three errors) and length 90 (two errors) appear** (Section 4). **Fourth, and this is the core.At length 90 the numbers fit, $1 + 90 + 4005 = 4096 = 2^{12}$, yet there is no code——the 88 weight-three words containing two given positions would have to be covered three at a time by weight-five codewords, and $88/3 = 29.333333$ is not an integer** (Section 4). **Fifth, a global count shows no contradiction——the 117480 weight-three words covered ten at a time by weight-five codewords need 11748 of them, a whole number. For the Golay code the local count also divides, $20/4 = 5$, and counting gives 5** (Section 4). **Sixth, the separator is whether one counts globally or locally with positions fixed——that the global numbers fit does not guarantee that the local numbers fit** (Section 5). **Keywords**: perfect code, Hamming bound, error-correcting code, Golay code, sphere packing.

Scope of this paper (scope note): **No new theorem or law is claimed——the Hamming codes, the Golay code, the sphere-counting bound (Hamming bound), and the counting argument that no perfect code of length 90 exists are all standard.Only binary codes are treated.The classification of perfect codes is not proved——that there are no binary perfect codes other than Hamming codes, the Golay code and trivial ones is cited, not proved.The search is limited——lengths meeting the number condition were sought up to 20000 and up to 8 errors corrected.Decoding procedures are not treated.Relation to earlier papers**: **Paper 315 showed that which dimension is special depends on the question, and stated in its scope note: "coding theory is not entered; the correspondence between sphere packing and error-correcting codes is not treated"——this paper takes that seat and counts how spheres pack in the space of words.Paper 281 showed that the quantum Hamming bound holds with equality at length 5, that is, that a quantum perfect code exists——this paper looks at the classical side, at a case where the numbers for equality fit and yet no code exists.Paper 163 showed that "a good code exists" and "here is a good code" are different sentences——this paper adds that "the numbers fit" and "it exists" are different sentences too.Paper 300 wrote that numerical agreement alone is not taken as sharing a root——the number condition here likewise does not guarantee existence by agreement alone; it is a caution of the same shape.**What is added is **checking exhaustively that the Hamming and Golay codes fill the space of words exactly**, **searching for lengths that meet the number condition**, **setting side by side that at length 90 the global count divides while the local count does not**, and **placing the separator on global versus local counting**.

1. Notation and the Question

Setting: **from the 2^n binary words of length n, choose 2^k as codewords.If any two codewords differ in at least $2e+1$ places, up to e errors can be corrected by returning to the nearest codeword.The spheres of radius e around the codewords do not overlap, so**

$$2^k \sum_{i=0}^{e} \binom{n}{i} \le 2^n,$$

and when equality holds every word lies in exactly one sphere. That is a perfect code.

Observation 1.1

for equality, $\sum_{i \le e} \binom{n}{i}$ must be a power of 2.That is a number condition, easily checked——whether a code exists whenever the numbers fit is another question.

Question: **when the points of the spheres divide the whole exactly, must the perfect code exist?**

2. The Hamming Code

Computation 2.1: the Hamming code of length 7 (parity-check columns the binary numbers 1 to 7) was built exhaustively.It has 16 codewords, the minimum weight of a non-zero codeword is 3, radius-one spheres have 8 points, and $16 \times 8 = 128 = 2^{7}$.Counting, for each of the 128 words, the codewords within distance 1 gives a minimum and a maximum of 1.For Hamming codes of length $2^{r} - 1$, $2^{n-r}(1+n) = 2^{n}$ holds for each of $r = 3, 4, 5, 6$.

> **Observation 2.2**
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> the local count divides too.There are 6 weight-two words containing a given position, covered by the weight-three codewords containing that position, two each; $6/2 = 3$ divides, and counting gives 3——the numbers fit globally and locally.

3. The Golay Code

Computation 3.1: the Golay code of length 23 was built from the generator polynomial $x^{11} + x^{10} + x^{6} + x^{5} + x^{4} + x^{2} + 1$.

weight	0	7	8	11	12	15	16	23
codewords	1	253	506	1288	1288	506	253	1

There are 4096 codewords of minimum weight 7; radius-three spheres have $1 + 23 + 253 + 1771 = 2048$ points, and $4096 \times 2048 = 8388608 = 2^{23}$.

> **Observation 3.2**
>
> for the Golay code, too, the local count divides.There are 20 weight-four words containing three given positions, covered four at a time by weight-seven codewords; $20/4 = 5$ divides, and counting gives 5——the global count $\binom{23}{4} / \binom{7}{4} = 253$ also matches the 253 codewords of weight 7.

4. The Number Condition and Length 90

Computation 4.1: pairs (n, e) with $\sum_{i \le e} \binom{n}{i}$ a power of 2 were sought for e from 2 to 8 and n up to 20000 (excluding trivial repetition codes $n = 2e + 1$).Only $(n, e) = (23, 3)$ and $(90, 2)$ appear; the first is the Golay code.The second meets the number condition: $1 + 90 + 4005 = 4096 = 2^{12}$.

Computation 4.2: suppose a perfect code of length 90 correcting two errors exists, and contains 0.There are 88 weight-three words with 1 in the first two positions.Each lies within distance 2 of exactly one codeword, which must have weight 5 and contain its three positions.A weight-five codeword containing the first two positions covers exactly 3 such words, one for each choice of its remaining three positions.So 88 must be divisible by 3, but $88/3 = \mathbf{29.333333}$, remainder 1.

> **Observation 4.3**
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> this is the core.At length 90 the number condition holds, yet there is no perfect code——that the points of the spheres divide the whole exactly does not guarantee that the spheres can actually be packed.

> **Observation 4.4**
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> a global count shows no contradiction.There are $\binom{90}{3} = 117480$ weight-three words in all, each weight-five codeword covering $\binom{5}{3} = 10$, so 11748 weight-five codewords would do, a whole number——the contradiction appears only when two positions are fixed.

5. The Separator

> **Observation 5.1**
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> the separator is whether one counts globally or locally with positions fixed.The Hamming and Golay codes fit both globally and locally; length 90 fits globally and not locally——the global number condition is necessary, and the local count is a stronger necessary condition.

Observation 5.2

taking the seat left by Paper 315.It showed that which dimension is special depends on the question, and set aside the correspondence between sphere packing and error-correcting codes in its scope note——this paper counts how spheres pack in the space of words at lengths 7, 23 and 90.

Observation 5.3

beside Paper 281 and Paper 163.Paper 281 showed that the quantum Hamming bound holds with equality at length 5; Paper 163 showed that "a good code exists" and "here is a good code" are different sentences——on the classical side, this paper shows at length 90 that "the numbers fit" and "it exists" are different sentences too.

Observation 5.4

a caution of the same shape as Paper 300.Paper 300 wrote that numerical agreement alone is not taken as sharing a root——the number condition here likewise does not guarantee a code by agreement alone. That the numbers fit is a reason to start the next count, not a conclusion.

Observation 5.5

what is not said.That the binary perfect codes are only the Hamming codes, the Golay code and trivial ones is cited, not proved; the search stopped at length 20000——what is shown is only that the number condition and existence are separate, counted out in one example.

6. Roll Call and Closing

1. **the length-7 Hamming code: $16 \times 8 = 128$** (Computation 2.1): **every word covered exactly once.**
2. **locally too, $6/2 = 3$** (Observation 2.2): **counting gives 3.**
3. **the Golay code: $4096 \times 2048 = 2^{23}$** (Computation 3.1): **minimum weight 7.**
4. **the Golay code's local count $20/4 = 5$** (Observation 3.2): **counting gives 5.**
5. **the number condition is met by $(23, 3)$ and $(90, 2)$** (Computation 4.1): **up to length 20000.**

6. **at length 90, $1 + 90 + 4005 = 2^{12}$** (Computation 4.1): **the numbers fit.**
7. **locally, $88/3 = 29.333333$** (Computation 4.2): **not whole; the core.**
8. **globally, $117480/10 = 11748$** (Observation 4.4): **whole.**
9. **taking the seat left by Paper 315** (Observation 5.2): **sphere packing and codes.**
10. **the separator is global versus local** (Observation 5.1): **fitting numbers do not guarantee existence.**

Closing: **Among error-correcting codes there are perfect codes, in which every word lies near exactly one codeword; their number condition is that the points of the spheres divide the whole exactly.The Hamming and Golay codes meet the condition and exist——the length-7 Hamming code covers the 128 words exactly once with 16 spheres of 8 points, and the length-23 Golay code fills 2^{23} with 4096 spheres of 2048 points; for both, the local count with positions fixed also divides.At length 90 the number condition holds, yet there is no code——$1 + 90 + 4005 = 4096 = 2^{12}$ fits, but the 88 weight-three words containing two given positions would have to be covered three at a time by weight-five codewords, and $88/3 = 29.333333$ is not whole. Counted globally, $117480/10 = 11748$ is whole and shows no contradiction.The separator is whether one counts globally or locally with positions fixed——that the global numbers fit guarantees neither that the local numbers fit nor that the code exists.Placed among the earlier papers——this takes the seat for sphere packing and codes that Paper 315 left in its scope note. Beside the quantum perfect code of Paper 281 and the distinction between "exists" and "here it is" in Paper 163, it shows the distinction between "the numbers fit" and "it exists". It is a caution of the same shape as Paper 300's refusal to take numerical agreement alone as sharing a root.To be honest——the classification of binary perfect codes is cited, not proved; the search stopped at length 20000 and 8 errors; decoding is not treated.**

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