The Window of Angles at Which a Gem Returns Light Upward Opens at Refractive Index $\sqrt{2}$ and Is Widest Near 2.3048

— the window opens at refractive index $\sqrt{2} = \mathbf{1.414214}$ and is widest, $\mathbf{12.857143}$ degrees, at $\mathbf{2.304765}$ — diamond's window of $\mathbf{12.2199}$ degrees is slightly narrower than cubic zirconia's $\mathbf{12.6902}$, and water has none — [Paper 598]

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— From thinking to syncing. —

Abstract

Diamond is said to sparkle because its refractive index is high.What this paper shows is that there is a window of pavilion angles at which light entering from straight above is returned upward by two total reflections, and that this window opens at refractive index $\sqrt{2}$, is widest at 2.304765, and narrows beyond.No new theorem or law is claimed. First, five conditions fix the window——total reflection at the right face, travelling towards the left face, total reflection at the left face, leaving through the top, and not tilting too far to the other side on the way out (Section 2). **Second, the window opens at refractive index $\sqrt{2}$——below a critical angle of 45 degrees, refractive index $\sqrt{2} = \mathbf{1.414214}$, there is no window. Water (1.333) has none; glass (1.5) has $\mathbf{4.2529}$ degrees and quartz (1.544) 6.1786 degrees** (Section 3). **Third, and this is the core.The window is widest, $\mathbf{12.857143}$ degrees, at refractive index $\mathbf{2.304765}$ (critical angle $180/7$ degrees), and beyond that narrows to half the critical angle——diamond's $\mathbf{12.2199}$ degrees is slightly narrower than cubic zirconia's $\mathbf{12.6902}$ degrees** (Section 3). **Fourth, tracing rays one by one, the window's edges agree with the closed form to within 0.0005 degrees——but the closed form first written left out the fifth condition, and diamond's upper edge was off by 0.7435 degrees; checking by two routes found it** (Section 4). **Fifth, Tolkowsky's 40.75 degrees lies in diamond's window and not in glass's——the same angle returns light or not according to the refractive index** (Section 4). **Sixth, the separator is which condition fixes the edges of the window ——at low index the total-reflection conditions fix the edges, at high index the conditions for leaving through the top, and where they exchange the window is widest** (Section 5). **Keywords**: gem cutting, total internal reflection, critical angle, refractive index, diamond.

Scope of this paper (scope note): **No new theorem or law is claimed——total reflection, the critical angle, and Tolkowsky's way of following rays in a cross-section are all standard.Only light entering from straight above is considered——oblique light, refraction at the crown facets, the variation of index with colour (fire) and absorption are not treated.Rays are followed by direction only, in a two-dimensional cross-section——which face a ray meets is decided by its direction; face sizes and points of incidence are not included.Refractive indices are representative.No gem is assessed and no cut is recommended.Relation to earlier papers**: **Paper 474 showed that on nights when distant sounds carry, the air above is warm and sound bends back down——the total reflection that returns sound in air and the total reflection that returns light in a stone come from the same critical-angle formula $\sin\theta_c = 1/n$; as two total reflections met in separate places, they share a root in the sense of Paper 300.Paper 300 showed that sharing a root is decidable——the three cases of the window width here are cases of one formula, and are not called sharing a root.**What is added is **writing the window with five conditions and its width in closed form over three ranges of index**, **finding the index $\sqrt{2}$ at which it opens and 2.304765 at which it is widest**, **checking the closed form by ray tracing and finding what the first form left out**, and **placing the separator on which condition fixes the edges**.

1. Notation and the Question

Setting: **take a cross-section of a gem with a horizontal top face (the table) and two lower faces (the pavilion) tilted at angle θ.Light entering from straight above reaches the viewer if it reflects at the right face, then at the left face, and leaves through the top.With critical angle θ_c ($\sin\theta_c = 1/n$), the conditions are**

$$\theta > \theta_c, \qquad \theta > 30^\circ, \qquad 180^\circ - 3\theta > \theta_c, \qquad 180^\circ - 4\theta < \theta_c, \qquad 4\theta - 180^\circ < \theta_c$$

(total reflection at the right face, heading for the left face, total reflection at the left face, leaving through the top, not tilting too far to the other side).

Observation 1.1

diamond is said to sparkle because its refractive index is high.The higher the index, the smaller the critical angle and the easier total reflection becomes——whether a higher index always returns light more readily is another question.

Question: **how does the window of pavilion angles that return light upward change with refractive index?**

2. Five Conditions

> **Observation 2.1**
>
> **two of the five conditions fix the edges of the window.The lower edge is the later of “total reflection at the right face” and “leaving through the top”; the upper edge is the earlier of “total reflection at the left face” and “not tilting too far”——the 30-degree condition never binds within the window.**

3. How the Window Opens

Computation 3.1: **the window for six materials, from the closed form and from ray tracing in steps of 0.0005 degrees.**

material	index	critical angle	window	width
diamond	**2.417**	**24.4397** deg	**38.8901**--**51.1099** deg	**12.2199** deg
cubic zirconia	**2.15**	**27.7177** deg	**38.0706**--**50.7608** deg	**12.6902** deg
sapphire	**1.77**	**34.4003** deg	**36.3999**--**48.5332** deg	**12.1333** deg
quartz	**1.544**	**40.3660** deg	**40.3660**--**46.5447** deg	**6.1786** deg
glass	**1.5**	**41.8103** deg	**41.8103**--**46.0632** deg	**4.2529** deg
water	**1.333**	**48.6066** deg	none	**0**

The width has a closed form in three ranges of index: $(180 - 4\theta_c)/3$ for $\sqrt{2} < n < 1.701302$ (critical angle 36 to 45 degrees), $(180 - \theta_c)/12$ for $1.701302 < n < 2.304765$ (critical angle $180/7$ to 36 degrees), and $\theta_c/2$ for $n > 2.304765$.At index 1.40 and 1.41 the width is 0, at 1.42 it is 0.3107 degrees and at 1.45 1.8629 degrees.

> **Observation 3.2**
>
> **the window opens at refractive index $\sqrt{2} = 1.414214$.With a critical angle above 45 degrees, an angle that gives total reflection at the right face cannot give it at the left——inside water, light cannot be returned upward by two reflections.**

Observation 3.3

this is the core.The window is widest, $\mathbf{12.857143}$ degrees, at refractive index $\mathbf{2.304765}$ (critical angle $180/7 = 25.714286$ degrees), and beyond that narrows to half the critical angle——with too high an index, light returning upward after two reflections tilts too far to the other side at the table and is totally reflected there. Diamond's 12.2199 degrees is slightly narrower than cubic zirconia's 12.6902 degrees.

4. Tracing Rays

Computation 4.1: **as a route separate from the closed form, the direction of light was reflected face by face, and the window's edges read in steps of 0.0005 degrees.For all six materials the traced edges agreed with the closed form to within 0.0005 degrees.**

Observation 4.2

the closed form first written left out the fifth condition.Setting the upper edge by "total reflection at the left face" alone, $(180 - \theta_c)/3 = 51.8534$ degrees, put diamond's upper edge 0.7435 degrees above the traced 51.1099 degrees——the condition $4\theta - 180 < \theta_c$, that light after two reflections must not tilt too far to the other side to leave, was missing. Without checking by two routes, diamond's window would have been written 0.7435 degrees too wide.

Observation 4.3

Tolkowsky's 40.75 degrees lies in diamond's window and not in glass's.Glass's window begins at 41.8103 degrees, so at 40.75 degrees there is no total reflection at the right face——the same angle does not return light at a different index. At 45 degrees the returned light leaves straight up (tilt 0 degrees).

Observation 4.4

a high index gathers incoming light close to the vertical.Light entering from air is tilted inside by no more than the critical angle: 24.4397 degrees in diamond, 41.8103 degrees in glass——this paper looked only at light from straight above, but in diamond oblique light too is bent close to the vertical.

5. The Separator

Observation 5.1

the separator is which condition fixes the edges of the window.At low index the lower edge is fixed by total reflection at the right face and the upper by total reflection at the left; at high index both edges are fixed by the conditions for leaving through the top——where they exchange (index 2.304765) the window is widest. "A higher index returns light more readily" holds only up to there.

Observation 5.2

one root with Paper 474, in the sense of Paper 300.It showed that on nights when distant sounds carry, the air above is warm and sound bends back——the total reflection that returns sound in air and the one that returns light in a stone are two met in separate places, yet come from the same critical-angle formula $\sin\theta_c = 1/n$.

Observation 5.3

what is not said.No gem is assessed and no cut recommended; oblique light, crown refraction, the variation of index with colour and absorption are not included——what is shown is that the window of angles that return light from above does not widen steadily with index, but is widest at a certain index.

6. Roll Call and Closing

1. **five conditions fix the window** (Notation): **two of them set its edges.**
2. **the window opens at index $\sqrt{2} = 1.414214$** (Observation 3.2): **water has none.**
3. **glass has 4.2529 degrees** (Computation 3.1): **quartz 6.1786.**
4. **widest, 12.857143 degrees, at index 2.304765** (Observation 3.3): **the core.**
5. **diamond 12.2199 degrees, cubic zirconia 12.6902 degrees** (Computation 3.1): **diamond is narrower.**
6. **the width in closed form over three ranges** (Computation 3.1): **$(180 - 4\theta_c)/3$, $(180 - \theta_c)/12$, $\theta_c/2$.**
7. **tracing and closed form agree to 0.0005 degrees** (Computation 4.1): **two routes.**
8. **the first form left out the fifth condition, off by 0.7435 degrees** (Observation 4.2): **found by tracing.**

9. **Tolkowsky's 40.75 degrees lies in diamond's window** (Observation 4.3): **not in glass's.**
10. **the separator is which condition fixes the edges** (Observation 5.1): **sharing a root with the sound of Paper 474.**

Closing: **Diamond is said to sparkle because its refractive index is high.There is a window of pavilion angles that return light from straight above by two total reflections——it is fixed by five conditions: total reflection at the right face, heading for the left face, total reflection at the left face, leaving through the top, and not tilting too far. Below refractive index $\sqrt{2} = 1.414214$ there is no window, and in water light cannot be returned by two reflections; glass has a window of 4.2529 degrees.The window is widest, 12.857143 degrees, at index 2.304765, and narrows beyond——diamond's 12.2199 degrees is slightly narrower than cubic zirconia's 12.6902 degrees, because with too high an index the returning light tilts too far to the other side at the table and cannot leave. Tracing rays one by one, the edges agreed with the closed form to within 0.0005 degrees, but the form first written left out this fifth condition and put diamond's upper edge 0.7435 degrees too high.The separator is which condition fixes the edges ——the window is widest where the fixing conditions exchange, and "a higher index returns light more readily" holds only up to there.Placed among the earlier papers——the total reflection of sound in air in Paper 474 and the total reflection of light in a stone come from the same critical-angle formula; as two met in separate places, they share a root in the sense of Paper 300.To be honest——only light from straight above was followed, by direction, in a two-dimensional section, without oblique light, crown refraction, colour dependence or absorption. The indices are representative; no gem is assessed and no cut is recommended.**

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