

c as the Ball of Velocity Space

— 2nd ed.: This Was a Result About c , Not About 4π (the Yamagishi Patch and Thomas Precession) — [Paper 60]

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— *From thinking to syncing.* —

ABSTRACT

In special relativity the set of admissible velocities is a ball of radius c , and under the metric induced by the velocity-addition law it is **hyperbolic space of curvature $K = -1/c^2$** (Varičák 1910). **Here the reading of c as a radius holds literally, not as a patch.** That curvature is **measured, as Thomas precession — the holonomy of velocity space.** On this known geometry the first edition rewrote the rest energy $E = 4\pi c^2$ of the $m = 4\pi$ representation (Paper 12) as the intrinsic quantity $E = 4\pi/|K|$. **What the second edition corrects is the standing of that 4π .** First, the 4π of $E = 4\pi c^2$ cannot be a solid angle. Since $[E] = ML^2T^{-2}$ while $[c^2] = L^2T^{-2}$, the factor in front **must carry a mass.** A solid angle is dimensionless (mass dimension $\mu = 0$) and cannot stand there. Under the patch $m = 4\pi$, this 4π is the mass itself (Paper 116: $\varepsilon = \mu$) — that is, **a convention, not a geometry** (Paper 56, 2nd ed.). **Second, $E = 4\pi/|K|$ requires the same reading.** $[1/|K|] = [c^2] = L^2T^{-2}$ is a squared velocity, not an area. Read as a solid angle, $4\pi/|K|$ would be a squared velocity and not an energy. The first edition's gloss — “ 4π is the solid angle, a fixed geometric constant” — **is not available** (Observation 3.3). **Third, the first edition's Section 5 therefore needs three entries, not two.** It rightly separated “the solid-angle 4π ” from “the topological $4\pi = 2\pi|\chi|$ ”; but **what stands in $E = 4\pi c^2$ is neither. It is a third thing — the mass patch, a convention** (Observation 5.2). **Fourth, and what remains is stronger than the first edition supposed.** The core is the geometric identity $c^2 = 1/|K|$, and it needs no patch whatever. Hence $E = mc^2 = m/|K|$ holds for any m . The 4π contributes nothing geometric; it merely **names the unit of mass.** **This paper's result was about c , not about 4π** (Corollary 3.5). **And its strongest point — that the curvature is measured as Thomas precession — stands untouched** (Section 4).

Scope of this paper: NO NEW PHYSICAL LAW AND NO NEW MATHEMATICAL THEOREM IS CLAIMED. That velocity space is hyperbolic (Varičák 1910), Thomas precession (Thomas 1926), the area of a hyperbolic geodesic sphere, and Gauss--Bonnet are all standard. What the second edition adds is only **a judgement about which 4π appears in the formula.**

1. Notation and position

Position: This paper claims no new physical law and no new mathematical theorem. Its contribution is to rewrite $E = 4\pi c^2$ (Paper 12) on this known geometry **as a coordinate-independent intrinsic quantity**, and to **separate what is intrinsic and which 4π has which origin**.

Added in the second edition: The first edition separated the origins of 4π into two. **The second edition shows that what actually stands in the formula is a third.**

2. Velocity space is hyperbolic with curvature $-1/c^2$ (known)

Proposition 2.1 (the hyperbolic geometry of velocity space; Varičák 1910)

In special relativity the physical velocities form the ball $\{v \in \mathbb{R}^3 : |v| < c\}$ of radius c . Under the metric induced by the velocity-addition law,

$$ds^2 = c^2 [d\varphi^2 + \sinh^2 \varphi (d\theta^2 + \sin^2 \theta d\psi^2)], \quad \varphi = \operatorname{artanh} \frac{v}{c} \quad (1)$$

this space is **hyperbolic three-space of sectional curvature $K = -1/c^2$** . The boundary $|v| = c$ corresponds to hyperbolic infinity (light).

Observation 2.2 (second edition: rapidity is not the distance)

The first edition said that “the rapidity φ equals the hyperbolic distance from the centre”. φ is **dimensionless**, and the distance from the centre in (1) is $r = c\varphi$. Indeed the area formula the first edition uses in Section 3 reads $\sinh^2(r/c) = \sinh^2 \varphi$ precisely because $r = c\varphi$. **The first edition's text disagreed with its own area formula by one factor of c** . The second edition corrects it.

What matters here is that c is at once a coordinate radius and an intrinsic radius of curvature:

$$K = -1/c^2 \iff c^2 = 1/|K|.$$

3. The intrinsic expression of $E = 4\pi c^2$, and the standing of its 4π

Proposition 3.1 (corroboration as an area scale, unchanged)

In hyperbolic three-space of curvature $-1/c^2$, the area of a geodesic sphere of radius r is

$$A(r) = 4\pi c^2 \sinh^2\left(\frac{r}{c}\right) \tag{2}$$

reducing to $4\pi r^2$ as $r \rightarrow 0$ (verified here by series expansion). So $4\pi c^2$ is the intrinsic area scale set by the curvature, not a coordinate-dependent quantity. This proposition is correct and unchanged in the second edition.

Proposition 3.2 (the intrinsic expression of E)

The $m = 4\pi$ representation ($E = 4\pi c^2$, Paper 12) may be written using the intrinsic curvature of velocity space as

$$E = 4\pi c^2 = \frac{4\pi}{|K|} \tag{3}$$

where $1/|K|$ is coordinate-independent.

3.1 [Second edition] This 4π is not a solid angle

Observation 3.3 (the dimensions decide)

The first edition glossed (3) with “ 4π is the solid angle, a fixed geometric constant”. That reading is not available.

Quantity	Dimension	Mass dimension μ
E	ML^2T^{-2}	1
$c^2, 1/ K $	L^2T^{-2}	0
solid angle $\int_{S^2} d\Omega$	dimensionless	0
mass patch $m = 4\pi$	M	1

Since E carries mass dimension 1 while c^2 and $1/|K|$ carry 0, the factor in front must carry mass dimension 1. A solid angle, being dimensionless, cannot stand there. So the 4π of $E = 4\pi c^2$ is the mass itself — as Paper 116 showed, the power of the patch constant is exactly the mass dimension ($\epsilon = \mu$).

Observation 3.4 ($1/|K|$ is not an area)

Likewise $[1/|K|] = [c^2] = L^2 T^{-2}$ is a **squared velocity**, not an area. The first edition's reading, “the solid angle times the squared intrinsic radius of curvature of velocity space”, contains **two misplacements at once**: it makes the left factor dimensionless and calls the right factor an area.

The $4\pi c^2$ of Proposition 3.1 is an area — because there the 4π (a solid angle) multiplies a squared **distance**. The $4\pi c^2$ of (3) is an energy — because there the 4π (a mass) multiplies a squared **velocity**. The same string of symbols denotes different quantities in two sections of one paper.

This is the shape Paper 51 exhibited for the shell theorem: the 4π surviving in the answer was the convention, not the geometry.

3.2 [Second edition] What remains — stronger than the first edition supposed

Corollary 3.5 (the core is $c^2 = 1/|K|$, and it needs no patch)

Strip the 4π from this section and what remains is

$$c^2 = \frac{1}{|K|} \quad \implies \quad E = mc^2 = \frac{m}{|K|} \quad (\text{for any } m)$$

The geometric identity that c^2 is the squared radius of curvature of velocity space is the core of this paper, and it needs no patch whatever. The 4π contributes nothing geometric; it merely names the unit of mass.

This paper's result was therefore about c , not about 4π — and it is the more robust for it, holding also for a reader who does not adopt the convention $m = 4\pi$.

Corollary 3.6 (energy as the reciprocal of the curvature, restated)

From $E = m/|K|$, the energy scale is proportional to **the reciprocal of how curved velocity space is**. In the limit where velocity space flattens ($c \rightarrow \infty$, the Newtonian limit, $|K| \rightarrow 0$) the energy scale diverges. **This reading holds whatever m may be.**

4. The curvature is measured: Thomas precession (unchanged)

Proposition 4.1 (Thomas precession as the holonomy of velocity space)

Parallel-transporting a spin around a closed circuit in velocity space (accelerate, turn, decelerate) produces a net rotation — **the Thomas--Wigner rotation**. Its angle equals **the magnitude $|K|$ of the curvature times the enclosed hyperbolic area** (the angle defect).

(A small correction in the second edition) The first edition wrote “the product of K and the area”; since K is negative, what equals the defect is $|K|$ **times the area**. The substance is unchanged.

The curvature of velocity space is thus not a geometric picture but a measured quantity — the Thomas factor $1/2$ in atomic spin-orbit coupling being its manifestation.

This section is the paper's strongest point and is not altered in the second edition. The reading “ c is the intrinsic radius of curvature of velocity space” stands on **measurable geometry**.

5. Discipline: which 4π , which dimension

Proposition 5.1 (separation from the topological 4π , unchanged)

For a closed surface of curvature $-1/c^2$, Gauss--Bonnet gives

$$\int K dA = 2\pi\chi = 2\pi(2 - 2g) \implies A = 4\pi c^2(g - 1)$$

so that $A = 4\pi c^2$ at genus $g = 2$ (verified here). **That $4\pi = 2\pi|\chi|$ originates in the Euler characteristic and has a different root from the solid angle** (Papers 48, 67). **Numerical agreement must not be taken for identity.**

5.1 [Second edition] Two entries are not enough

Observation 5.2 (there is a third — and it is the one in the formula)

The first edition separated the origins of 4π into **the solid angle** and **the topological**. **The separation is right, but the census is short.**

4π	Origin	μ	Standing
solid angle $\int_{S^2} d\Omega$	the geometry of the sphere (Paper 01)	0	fact
$2\pi \chi $ at $g = 2$	the Euler characteristic (Paper 67)	0	fact
mass patch $m = 4\pi$	a choice of units (Papers 56, 116)	1	convention

And what stands in $E = 4\pi c^2$ is the third (Observation 3.3). The first edition carefully separated two facts and overlooked the convention actually standing in its formula.

The 4π of $A(r) = 4\pi c^2 \sinh^2(r/c)$ in Proposition 3.1 is the first. Within one paper, the first and the third were written with the same symbol — the source of the first edition's confusion.

A note on dimension (unchanged): Velocity space has the same dimension as position space, namely three. So the $d = 3$ here is a different dimension by a different criterion from the $d = 5$ of Paper 59. The second edition of Paper 59 has since shown that its $d = 5$ follows from setting the radius to one (a convention), while also stating that reading c as a radius holds literally for the velocity ball — the latter being the same content as Proposition 2.1 here.

6. Conclusion

1. Velocity space is a ball of radius c and is hyperbolic with curvature $K = -1/c^2$ (Proposition 2.1; Varičák 1910). Here the reading of c as a radius holds literally.
2. [2nd ed.] Rapidity is not the distance — φ is dimensionless and the distance is $c\varphi$. The first edition's text disagreed with its own area formula by one factor of c (Observation 2.2).
3. $A(r) = 4\pi c^2 \sinh^2(r/c)$ reduces to $4\pi r^2$ as $r \rightarrow 0$, corroborating $4\pi c^2$ as the intrinsic area scale set by the curvature (Proposition 3.1, unchanged).
4. [2nd ed.] The 4π of $E = 4\pi c^2$ is not a solid angle — since $[E]$ has mass dimension 1 while $[c^2]$ and $[1/|K|]$ have 0, the factor in front must be a mass, and a solid angle, being dimensionless, cannot stand there. This 4π is the mass patch, that is a convention (Observations 3.3, 3.4). It is the shape of Paper 51.
5. [2nd ed.] The census of 4π therefore needs three entries — solid angle (fact), topological $2\pi|\chi|$ (fact), mass patch (convention). What stands in $E = 4\pi c^2$ is the third; what stands in $A(r)$ is the first. One symbol denoted two quantities within one paper (Observation 5.2).
6. [2nd ed.] What remains is stronger than the first edition supposed — the core is the geometric identity $c^2 = 1/|K|$, which needs no patch, and $E = mc^2 = m/|K|$ holds for any m . The result was about c , not about 4π (Corollaries 3.5, 3.6).

7. **The curvature is measured as Thomas precession** (Proposition 4.1, unchanged): the defect is $|K|$ times the enclosed hyperbolic area, manifested in the Thomas factor $1/2$ of atomic spin-orbit coupling. **This is the paper's strongest point and is not altered.**

Closing: The first edition carefully separated two origins of 4π and warned against confusing them. **The warning was right; only the count was short** — there was a third standing in the formula itself, the 4π that is a convention. Remove it, and **what is left is firmer than what was there:** c^2 is the squared radius of curvature of velocity space, and that curvature is measured as Thomas precession. **Taking the patch away does not always weaken a result; sometimes it strengthens it.**

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