No Latin Square of Order Six Has an Orthogonal Mate

— of the 9408 order-6 squares with first row and column in order, 0 have an orthogonal mate — many have transversals, yet none can be split into 6 disjoint transversals — [Paper 604]

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— From thinking to syncing. —

Abstract

A Latin square of order n places n symbols in an $n \times n$ grid so that each occurs once in every row and every column.Two Latin squares are orthogonal if, when superimposed, all n^2 pairs of symbols occur exactly once.Euler asked whether officers of six ranks from six regiments can stand so that every row and every column holds one officer of each regiment and each rank, which is to ask whether there is an orthogonal pair of order 6.No new theorem or law is claimed. First, with first row and column in order, there are 9408 squares of order 6——constructing all such squares gives $1, 1, 1, 4, 56, 9408$ for orders 1 to 6. The full numbers without this ordering are 576 for order 4 and 161280 for order 5, both agreeing with counts made by constructing them all (Section 2). **Second, and this is the core.None of the 9408 squares of order 6 has an orthogonal mate——having a mate is the same as splitting the square into n disjoint transversals (sets of n cells taking one from every row, every column and every symbol). All 9408 were checked, and 0 could be split** (Section 3). **Third, smaller orders have mates——the squares with a mate number 0 in order 2, 1 of 1 in order 3, 1 of 4 in order 4 and 6 of 56 in order 5. In orders 3 and 4 the answer by transversals and the answer by trying every candidate mate agreed square by square** (Section 3). **Fourth, the lack of a mate is not a lack of transversals——of the order-6 squares only 2100 (0.223214) have no transversal; the rest have 8 (7020 squares), 24 (108) or 32 (180). Still, none can be split into 6 disjoint ones** (Section 3). **Fifth, the separator is not "even" but "six"——the square with $i + j$ modulo n in row i and column j has 0 transversals in even orders (2, 4, 6) and 3, 15, 133 in odd orders. But in order 4 another square, the table of the Klein four-group, has 8 transversals and a mate. It is not that even squares all lack mates; only order 6 lacks a mate for every square** (Section 4). **Keywords**: Latin square, orthogonal Latin squares, thirty-six officers problem, transversal, combinatorics.

> **Scope of this paper (scope note)**: **No new theorem or law is claimed——Euler's question (1782), that there is no orthogonal pair of order 6 (Tarry, 1900), that Euler's conjecture fails from order 10 on (Bose, Shrikhande and Parker, 1960), and the numbers of small Latin squares (McKay, Meynert and Myrvold, 2007) are all known.Tarry's proof is not reproduced——the 9408 squares with first row and column in order are constructed and checked one by one by transversals.Orders 10 and above are not computed——the side on which Euler's conjecture (no pair for orders leaving remainder 2 on division by 4) fails is stated only as a known result.Uses in the design of experiments are not treated.Relation to earlier papers**: **Paper 247 counted eight ways of "splitting into even and odd"——that cyclic squares of even order have no transversal is one split by parity, but the absence in order 6 cannot be explained by it.**What is added is **giving the distribution of transversal counts over the 9408 ordered squares of order 6, matching the transversal method with exhaustive trial square by square in orders 3 and 4**, and **placing the separator on "six" rather than "even" and checking it by setting the cyclic squares of even order beside the Klein four-group table of order 4**.

1. Notation and the Question

Setting: **a Latin square L of order n places the symbols $0, 1, \ldots, n-1$ in an $n \times n$ grid so that each occurs once in every row and every column.Two squares L, M are orthogonal if all n^2 pairs (L_{ij}, M_{ij}) occur exactly once.A transversal of L is a set of n cells, one from every row and every column, whose symbols are all different.**

Mates and transversals: **if M is orthogonal to L, the n cells where M has the same symbol form a transversal of L, and n of them split all the cells. Conversely, if L can be split into n disjoint transversals, placing one symbol per transversal gives M.**

Why ordering is allowed: **permuting the rows of L and M in the same way, or their columns in the same way, keeps them orthogonal.Every square can be brought, by permuting columns, to have first row $0, 1, \ldots, n-1$, and then, by permuting the other rows, to have first column $0, 1, \ldots, n-1$ too.So whether a mate exists can be decided on squares with first row and column in order (reduced squares) alone.**

Question: **does a square of order 6 have an orthogonal mate? If not, is that because it has no transversal, or because 6 is even?**

2. Constructing All the Squares

Computation 2.1: **all reduced squares were constructed by a search that fills one cell at a time.**

order	1	2	3	4	5	6
reduced squares	1	1	1	4	56	9408

Computation 2.2: constructing and counting all squares without the ordering gave 576 for order 4 and 161280 for order 5.These agree with $n!(n-1)!$ times the reduced number: $4!3! \times 4 = 576$ and $5!4! \times 56 = 161280$.By the same product, order 6 has $6!5! \times 9408 = 812851200$ squares in all.

> **Observation 2.3**
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> the two counts agree.The reduced squares and the full squares were each constructed separately, and the product relation held in orders 4 and 5——9408 is also the value given by McKay and coauthors.

3. Looking for Mates

Computation 3.1: for every reduced square, all transversals were counted, and whether it splits into n disjoint transversals (has a mate) was checked.

order	reduced squares	with a mate	transversals (number of squares with that many)
2	1	0	0 (1)
3	1	1	3 (1)
4	4	1	0 (3), 8 (1)
5	56	6	3 (50), 15 (6)
6	9408	0	0 (2100), 8 (7020), 24 (108), 32 (180)

The 6 squares of order 5 with a mate coincided exactly with the 6 having 15 transversals.

Computation 3.2: independently of the transversal method, in orders 3 and 4 every full square (12 of order 3, 576 of order 4) was superimposed to see whether any is orthogonal.The reduced squares with a mate were 1 in order 3 and 1 in order 4, agreeing square by square with the transversal method.

> **Observation 3.3**
>
> this is the core: in order 6 no square has a mate.The thirty-six officers cannot be arranged, however they are placed——this is what Tarry showed in 1900; this paper confirmed it by checking the 9408 reduced squares one by one.

Observation 3.4

the lack of a mate is not a lack of transversals.Only 2100 squares of order 6 (share 0.223214) have no transversal; 7308 have 8 or more, and the most is 32——there are enough transversals, yet 6 mutually disjoint ones cannot be picked out. The issue is not how many, but how they fit together.

4. The Separator

Computation 4.1: **transversals were counted for the square with the remainder of $i+j$ on division by n in row i and column j (the cyclic square).For orders 2 to 7 they were $0, 3, 0, 15, 0, 133$.**

Observation 4.2

cyclic squares of even order have no transversal.Orders 2, 4 and 6 give 0, while odd orders give 3, 15 and 133——so far it looks like a phenomenon split by even and odd.

Observation 4.3

the separator is not "even" but "six".The only reduced square of order 4 with a mate is the table adding i and j in binary without carries (the table of the Klein four-group), which has 8 transversals. A square other than the cyclic one has a mate even in an even order——order 4 has a square with a mate and order 6 has none. Apart from order 2, only order 6 lacks a mate for every square. Euler conjectured that this extends to every order leaving remainder 2 on division by 4, but it fails from order 10 on (Bose, Shrikhande and Parker).

Observation 4.4

beside Paper 247.Paper 247 counted eight ways of "splitting into even and odd"——that cyclic squares of even order have no transversal may be counted as one such split. But the absence in order 6 is not a split by parity; order 4 is even and has a mate.

Observation 4.5

what is not said.Tarry's proof is not reproduced, orders 10 and above are not computed, and uses in the design of experiments are not treated——what is shown is that none of the 9408 reduced squares of order 6 has a mate, the distribution of transversal counts, and that "even" is not the separator.

5. Roll Call and Closing

1. **reduced squares 1, 1, 1, 4, 56, 9408** (Computation 2.1): **constructed in full.**
2. **576 for order 4 and 161280 for order 5 match the product** (Computation 2.2): **two counts.**
3. **order 6 has 812851200 squares in all** (Computation 2.2): **9408 reduced ones suffice.**
4. **0 squares of order 6 with a mate** (Computation 3.1): **the core.**
5. **1, 1 and 6 squares with mates in orders 3, 4, 5** (Computation 3.1): **smaller orders have them.**
6. **transversals and exhaustive trial agree square by square in orders 3 and 4** (Computation 3.2): **two routes.**
7. **2100 squares of order 6 without a transversal (0.223214)** (Observation 3.4): **the rest have 8 or more.**
8. **transversals exist but do not split into 6** (Observation 3.4): **how they fit, not how many.**
9. **cyclic squares of even order have 0 transversals** (Computation 4.1): **it looks like a parity split.**
10. **the Klein four-group table of order 4 has a mate** (Observation 4.3): **the separator is "six".**

Closing: **Euler asked whether officers of six ranks from six regiments can stand so that every row and every column holds one of each regiment and each rank.No Latin square of order 6 has an orthogonal mate——constructing all 9408 squares of order 6 with first row and column in order, and checking each for a split into 6 disjoint transversals, none could be split. Orders 3, 4 and 5 have squares with mates, and in orders 3 and 4 the transversal method and exhaustive trial agreed square by square.The lack of a mate is not a lack of transversals——only 2100 squares of order 6 have none; most have 8 or more, up to 32, yet 6 disjoint ones cannot be picked out.The separator is not "even" but "six"——cyclic squares of even order have no transversal, but in order 4 the Klein four-group table has a mate.Placed among the earlier papers——beside the parity splits counted in Paper 247 it sets the absence in order 6, which parity cannot explain.To be honest——Tarry's proof was not reproduced, only confirmed by exhaustive trial of the reduced squares, and that Euler's conjecture fails from order 10 on was not computed.**

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