The Odd Numbers in Pascal's Triangle Are Counted by the Ones in the Row Number Written in Binary

— Row n holds 2^s odd numbers, where s is the number of ones in n written in binary — an entry is odd exactly when adding its two indices in binary produces no carry — [Paper 608]

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September 2026

— *From thinking to syncing.* —

Abstract

Pascal's triangle sets 1 at both ends of each row and makes each inner number the sum of the two above it. The k-th number from the left in row n (counting from 0) is the number of ways of choosing k things from n.How many of them are odd is counted.No new theorem or law is claimed. First, row n holds 2^s odd numbers——s is the number of ones in n written in binary. Counting with rows added modulo two, this held for every row from 0 to 1023. Row 255 (eight ones in binary) has all 256 entries odd, while row 256 (one one) has only the 2 at its ends (Section 2). **Second, the share of odd numbers tends to zero as rows are added——the first 2^k rows held 3^k odd numbers for every k from 0 to 10. Each doubling of the rows only triples the odd numbers, so the share falls to 0.750000 in 8 rows, 0.199447 in 256 and 0.112517 in 1024** (Section 3). **Third, and this is the core.An entry is odd exactly when adding k and $n-k$ in binary produces no carry——the exponent of the power of 2 dividing a number of choices equals the number of carries when k and $n-k$ are added in binary; computing the numbers of choices exactly, this held for every $n<256$. Choosing 2 of 6 gives 15, odd, since adding 2 and 4 carries nothing; choosing 3 of 6 gives 20, since adding 3 and 3 carries twice, divisible by 4** (Section 4). **Fourth, the separator is which prime one divides by ——the number of entries not divisible by 3 is the product of each base-3 digit of n plus one, which held for every row from 0 to 728. Divisibility by 3 is also decided by the number of carries in base 3** (Section 5). **Keywords**: Pascal's triangle, parity of binomial coefficients, Lucas's theorem, Kummer's theorem, binary representation.

Scope of this paper (scope note): **No new theorem or law is claimed——telling the parity of the numbers of choices by binary digits (Lucas, 1878), that the number of odd entries is 2^s (Glaisher, 1899), that the exponent of a prime dividing them is the number of carries (Kummer, 1852), and that the number of entries not divisible by a prime p is decided by base-p digits (Fine, 1947) are all known.The theory of the figure is not treated——of the pattern of odd entries (the Sierpinski triangle), only its self-similarity dimension is stated in one line.The theorems are not proved——rows up to 1023 (up to 728 modulo 3) are counted and matched with the exactly computed numbers of choices.Relation to earlier papers**: **Paper 602 showed that who wins at Nim is decided by addition without carries——here a number of choices being odd is the same as an addition producing no carry.**What is added is **matching the two routes of rows added modulo two and of the numbers of choices themselves**, **setting out in numbers how the share of odd entries falls over 2^k rows**, and **placing the separator on which prime one divides by and checking it modulo 3**.

1. Notation and the Question

Setting: **write the k-th number from the left in row n ($0 \le k \le n$) as $\binom{n}{k}$.The ends are 1 and each inner number is the sum of the two above.Let $s(n)$ be the number of ones in n written in binary.**

Question: **how many odd numbers does row n hold, which $\binom{n}{k}$ are odd, and what happens when dividing by 3 instead of 2?**

2. Odd Numbers Row by Row

Computation 2.1: **rows were built keeping only odd or even (modulo two), and the odd entries in each row were counted.**

row n	binary	odd entries	$2^{s(n)}$
6	110	4	4
7	111	8	8
8	1000	2	2
100	1100100	8	8
255	11111111	256	256
256	100000000	2	2
1023	1111111111	1024	1024

For every row from 0 to 1023, the number of odd entries equalled $2^{s(n)}$.

> **Observation 2.2**
>
> what matters is not the size of the row but the number of ones in its number.Row 255 has all 256 entries odd, yet the very next row 256 has only the 2 at its ends——255 has eight ones in binary, 256 has one.

3. The Share of Odd Numbers

Computation 3.1: the odd numbers in the first 2^k rows were counted.

first rows	odd	all entries	share
8	27	36	0.750000
16	81	136	0.595588
64	729	2080	0.350481
256	6561	32896	0.199447
1024	59049	524800	**0.112517**

For every k from 0 to 10, the first 2^k rows held 3^k odd numbers.

Observation 3.2

each doubling of the rows only triples the odd numbers.All entries grow roughly with the square of the number of rows, about fourfold, so the share of odd numbers is multiplied by about $3/4$ each time and tends to zero——the pattern of odd entries is self-similar, tripling its coloured area when enlarged twofold, with dimension $\log 3/\log 2 = 1.584963$.

4. Carries

Computation 4.1: computing $\binom{n}{k}$ exactly ($n < 256$), two things were checked.

First: $\binom{n}{k}$ is odd exactly when every binary digit where k has a 1 is also 1 in n.

Second: the exponent of the power of 2 dividing $\binom{n}{k}$ equals the number of carries when k and $n-k$ are added in binary.

Examples: $\binom{6}{2} = 15$: adding $2 = 010$ and $4 = 100$ gives 0 carries, and it is odd. $\binom{6}{3} = 20$: adding $3 = 011$ and $3 = 011$ gives 2 carries, and $20 = 2^2 \times 5$.

Observation 4.2

this is the core: odd means an addition with no carry at all.When k and $n-k$ add up to n in binary without carrying, the ones of k and of $n-k$ never meet in the same digit; they merely share out the ones of n——if n has s ones, the 2^s ways of handing each of them to one side or the other are the odd entries of row n.

Observation 4.3

beside Paper 602.Paper 602 showed that who wins at Nim is decided by addition without carries——a number of choices being odd is likewise decided by the absence of carries: it is odd exactly when the sum of k and $n-k$ without carries equals n.

5. The Separator

Computation 5.1: rows were built keeping only remainders on division by 3 ($n \le 728$), and entries not divisible by 3 were counted.

row n	base 3	not divisible by 3
5	12	6
8	22	9
26	222	27
27	1000	2
80	2222	81

For every row from 0 to 728, the number of entries not divisible by 3 equalled the product of each base-3 digit of n plus one. The first $3^6 = 729$ rows held $46656 = 6^6$ of them.The exponent of 3 also equalled the number of carries when adding k and $n - k$ in base 3 ($n < 244$).

> **Observation 5.2**
>
> **the separator is which prime one divides by.Dividing by 2 the binary digits matter; dividing by 3 the base-3 digits; in both, the number of carries decides the divisibility——binary digits are 0 or 1, so "plus one and multiply" gives 2^s; in base 3, a digit 2 contributes a factor 3.**

> **Observation 5.3**
>
> **what is not said.The theory of the Sierpinski triangle as a figure is not entered, and the theorems are not proved——what is shown is the count up to row 1023, the match with exactly computed numbers of choices, and the check modulo 3.**

6. Roll Call and Closing

1. **row n holds $2^{s(n)}$ odd numbers** (Computation 2.1): **true for rows 0 to 1023.**
2. **row 255 all odd, row 256 only 2** (Observation 2.2): **the binary ones matter.**
3. **the first 2^k rows hold 3^k odd numbers** (Computation 3.1): **$k = 0$ to 10.**
4. **share 0.112517 in 1024 rows** (Computation 3.1): **tending to zero.**
5. **self-similarity dimension 1.584963** (Observation 3.2): **three times per doubling.**
6. **oddness is decided by digits contained in digits** (Computation 4.1): **true for $n < 256$.**

7. **the exponent of 2 is the number of carries** (Computation 4.1): **15 has 0, 20 has 2.**
8. **odd means an addition without carries** (Observation 4.2): **the core.**
9. **entries not divisible by 3 follow base-3 digits** (Computation 5.1): **true to row 728.**
10. **the separator is which prime** (Observation 5.2): **binary for 2, base 3 for 3.**

Closing: **The numbers of Pascal's triangle are made as sums of the two above.Row n holds 2^s odd numbers, where s is the number of ones in n written in binary——counting with rows added modulo two, this held for every row from 0 to 1023; the first 2^k rows hold 3^k odd numbers, and the share falls to 0.112517 in 1024 rows.An entry is odd exactly when adding k and $n-k$ in binary produces no carry at all——the exponent of the power of 2 dividing it equals the number of carries, which held for all $n < 256$ when the numbers of choices were computed exactly. The 2^s ways of sharing the ones of n between k and $n-k$ are the odd entries.The separator is which prime one divides by——dividing by 3, the base-3 digits and carries matter, and this held up to row 728.Placed among the earlier papers——the addition without carries that Paper 602 found deciding Nim here decides the parity of the numbers of choices.To be honest——the theorems are not proved; rows up to 1023 were only counted and matched.**

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