The Worst Loss in Renting or Buying Shrinks When the Day of Purchase Is Left to Chance

— Deciding whether to rent skis for 1 a day or buy them for B without knowing how many days one will ski — the difference is whether the opponent knows one's choice — [Paper 610]

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— *From thinking to syncing.* —

Abstract

Skis can be rented for 1 a day or bought for B. How many days one will ski is not known until the skiing is over.The amount paid is compared with the least one could have paid knowing the number of days (renting throughout if under B days, buying if B or more), and the worst value of that ratio is called the worst loss.No new theorem or law is claimed. First, buying on a fixed day, the worst loss cannot go below $2 - 1/B$ times——for $B = 10$, trying every number of rental days before buying from 0 to 59, the worst loss was 10 times for 0 days, 2.5 for 5, the minimum 1.9 for 9, 2 for 10, and 4 for 30 (Section 2). **Second, leaving the day of purchase to chance lowers the worst loss——for $B = 10$, choosing the day of purchase among days 1 to 10 with probabilities from 0.059482 to 0.153534, heavier for later days, gives a worst loss of $\mathbf{1.535340}$ times. For $B = 2$ it is 1.333333, for $B = 3$ 1.421053, for $B = 100$ 1.577368, always below the fixed-day values 1.5, 1.666667 and 1.99** (Section 3). **Third, and this is the core.The separator is whether the opponent knows one's choice——buying on a fixed day, the opponent can end the skiing exactly on that day and make the loss largest. Left to chance, the opponent knows only the probabilities, and whatever number of days is chosen the loss ratio is the same 1.535340 (within 10^{-12} for every day from 1 to 10)** (Section 3). **Fourth, the formula and the search agree——the formula $1/(1 - (1 - 1/B)^B)$ for the worst loss agreed within 10^{-9} with the value obtained by evaluating the probabilities over all numbers of days for $B = 2, 3, 5, 10, 100$. For $B = 2$ and 3, searching the probabilities on a grid found no worst loss below the formula** (Section 4). **Fifth, for large B it approaches $e/(e - 1)$——because $(1 - 1/B)^B$ approaches $1/e$; in the grouping of the sources of $1/e$ made by Paper 479, it belongs to the second group, the limit from binomial to Poisson** (Section 4). **Keywords**: ski rental problem, competitive ratio, online algorithm, randomised algorithm, e/(e-1).

Scope of this paper (scope note): **No new theorem or law is claimed——that the fixed-day method gives $2 - 1/B$ times in the rent-or-buy problem (Karlin and coauthors, 1988), that leaving the day to chance approaches $e/(e-1)$ times (Karlin and coauthors, 1994), and measuring methods by their worst ratio against an opponent (Borodin and El-Yaniv, 1998) are all known.The case where the distribution of skiing days is known is not treated——only the worst case, in which the opponent chooses the number of days, is looked at.No actual purchase is recommended——skis are an example of a decision whose future is unknown.Relation to earlier papers**: **Paper 388 showed that in picking the best of candidates seen one by one (the secretary problem), the chance of success settles at $1/e$——this is also a decision with an unknown future, but what is measured is not the chance of success but the ratio to the best in hindsight.Paper 479 sorted the seven appearances of $1/e$ in the corpus into three groups by their source——the $e/(e-1)$ here comes from the limit of $(1-1/B)^B$, so it belongs to the second group.**What is added is **trying the fixed-day method on every day to confirm the best**, **matching the worst loss of the chance method by the formula, evaluation over all numbers of days, and a grid search**, and **placing the separator on whether the opponent knows one's choice, showing that under chance the loss ratio is the same whatever the number of days**.

1. Notation and the Question

Setting: **renting costs 1 a day and buying costs B (an integer). The number of skiing days d is not known until the skiing ends.The least one could pay knowing the number of days is $\min(d, B)$.With $C(d)$ the amount a method pays, define the worst loss as**

$$R = \max_{d \geq 1} \frac{C(d)}{\min(d, B)}.$$

The fixed-day method is "rent k days, and buy if skiing on day $k+1$"; the chance method chooses j in "buy on day j" with some probabilities.

Question: **how far can the worst loss be lowered, and what differs between a fixed day and chance?**

2. Buying on a Fixed Day

Computation 2.1: **for $B = 10$, the worst loss of renting k days before buying was found by evaluating $d = 1$ to 400 days.**

rental days k	worst loss
0 (buy on the first day)	10 times
5	2.5 times
8	2 times
9	1.9 times
10	2 times
15	2.5 times
30	4 times

Trying every k from 0 to 59, the smallest worst loss was 1.9 times at $k = 9$, equal to $2 - 1/B$.

> **Observation 2.2**
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> against a fixed day, the opponent ends the skiing right after the purchase.With $k = 9$, if skiing ends just after buying on day 10, one has paid $9 + 10 = 19$ against a best of 10: 1.9 times——buying early lets the opponent end after few days; buying late piles up rent. The balance is at $B - 1$ days, and even there nearly twice cannot be avoided.

3. Leaving It to Chance

Computation 3.1: the probability of buying on day j was made proportional to q^{B-j} with $q = (B-1)/B$ ($j = 1, \ldots, B$), and the worst loss was evaluated for $d = 1$ to 400 days.

B	worst loss by chance	formula $1/(1-(1-1/B)^B)$	best fixed day $2-1/B$
2	1.333333	1.333333	1.5
3	1.421053	1.421053	1.666667
5	1.487387	1.487387	1.8
10	**1.535340**	1.535340	1.9
100	1.577368	1.577368	1.99

For $B = 10$ the probabilities from day 1 on are $0.059482, 0.066091, 0.073435, 0.081594, 0.090660, 0.100734, 0.111926, 0.124363, 0.138181, 0.153534$, summing to 1.

Computation 3.2: for the chance method with $B = 10$, the loss ratio (expected payment over the best) for each number of days $d = 1$ to 10 was 1.535340 every time, with differences within 10^{-12}.

> **Observation 3.3**
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> this is the core: the separator is whether the opponent knows one's choice.Buying on a fixed day, the opponent knows the day and ends the skiing right after the purchase——left to chance, the opponent knows only the probabilities, and whatever number of days is chosen the loss ratio is the same. The probabilities are placed so that wherever the opponent strikes, the loss is no larger.

4. Formula and Search

Computation 4.1: independently of the formula, for $B = 2$ and 3 every probability on a grid of step $1/300$ was tried, looking for the smallest worst loss.

B	smallest on the grid	probabilities there	formula
2	1.333333	0.3333, 0.6667	1.333333
3	1.421667	0.21, 0.3167, 0.4733	1.421053

Both were within 0.01 of the formula, and no value below the formula was found.

Observation 4.2

for large B it approaches $e/(e-1) = 1.581977$.Because $(1-1/B)^B$ in the formula approaches $1/e$; at $B = 100$ it has reached 1.577368——for finite B it stays below the limit; 1.535340 at $B = 10$ is 0.046637 below it.

Observation 4.3

beside Paper 479 and Paper 388.Paper 479 sorted the seven appearances of $1/e$ in the corpus into a group from a stationarity condition, a group from the limit from binomial to Poisson, and one from how a function was written down. Paper 388 showed that the chance of success in the secretary problem settles at $1/e$—— the $1/e$ here comes from the limit of $(1-1/B)^B$ and so belongs to the second group of Paper 479. And as Paper 479 saw for the secretary problem, at finite B the value departs from its limit.

Observation 4.4

what is not said.The best way to buy when the distribution of skiing days is known is not treated, and no actual purchase is recommended——what is shown is the best fixed-day method, the worst loss of the chance method, and what separates the two.

5. Roll Call and Closing

1. **the best fixed day rents 9 days before buying** (Computation 2.1): **1.9 times $= 2 - 1/B$.**
2. **buying on day one gives 10 times, renting 30 days 4 times** (Computation 2.1): **too early or too late both lose.**
3. **the opponent ends the skiing right after the purchase** (Observation 2.2): **19 against 10.**
4. **chance gives 1.535340 times for $B = 10$** (Computation 3.1): **below 1.9.**
5. **chance is lower for every B** (Computation 3.1): **1.333333 against 1.5 for $B = 2$.**
6. **the same loss ratio for every number of days** (Computation 3.2): **within 10^{-12}.**
7. **the separator is whether the opponent knows one's choice** (Observation 3.3): **the core.**
8. **grid search finds nothing below the formula** (Computation 4.1): **two routes.**
9. **for large B, $e/(e-1) = 1.581977$** (Observation 4.2): **1.577368 at $B = 100$.**
10. **$1/e$ comes from the limit of $(1-1/B)^B$** (Observation 4.3): **the second group of Paper 479.**

Closing: **One decides whether to rent skis for 1 a day or buy them for B without knowing how many days one will ski.Buying on a fixed day, the worst loss cannot go below $2 - 1/B$ times——trying every day for $B = 10$, renting 9 days before buying is best at 1.9 times, and the opponent strikes by ending the skiing right after the purchase.Leaving the day of purchase to chance lowers the worst loss——choosing the day with probabilities heavier for later days gives 1.535340 times for $B = 10$, and the loss ratio was the same whatever number of days the opponent chose. The formula, evaluation over all days and a grid search agreed.The separator is whether the opponent knows one's choice——a fixed day is known; left to chance, only the probabilities are known.Placed among the earlier papers——the approach to $e/(e-1)$ for large B comes from $(1 - 1/B)^B$ approaching $1/e$, placing it in the second group of Paper 479, and it is a decision with an unknown future like the secretary problem of Paper 388.To be honest——the case where the distribution of skiing days is known is not treated; only the case where the opponent chooses the worst number of days is looked at.**

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