The Leading Digit Is 1 Three Times in Ten, but Only for Data Spread Across Many Orders of Magnitude

— Among 2^n for $n = 1$ to 10000 the leading digit is 1 in 0.301000 of cases, within 0.000092 of $\log_{10}(1 + 1/d)$ for all nine digits — the dividing line is whether the data spread across orders of magnitude — [Paper 612]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
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— *From thinking to syncing.* —

Abstract

Counting which of 1 to 9 is the leading digit (the top digit) in a list of numbers often gives about three in ten for 1 and under five per cent for 9 (Benford's law).For which data it holds, and for which it does not, is counted.No new theorem or law is claimed. First, powers of 2 follow Benford's law closely——computing 2^n exactly as integers for $n = 1$ to 10000 and counting leading digits, 1 appears 3010 times (0.301000) and 9 appears 458 times, within 0.000092 of $\log_{10}(1 + 1/d)$ for all nine digits. The first 5000 Fibonacci numbers also stay within 0.000347 (Section 2). **Second, multiplying everything by the same number leaves the distribution unchanged——multiplying all 2^n by 3 , the share with leading digit 1 is still 0.301000, with a largest difference of 0.000108** (Section 3). **Third, data that do not span orders of magnitude do not follow it——uniform numbers from 1 to below 10 give 0.110675 for leading digit 1 (close to $1/9$), and doubling them changes it to 0.554595. Heights averaging 170 cm give almost all (0.999985) leading digit 1** (Section 4). **Fourth, and this is the core.The separator is whether the data spread across orders of magnitude——products of 20 uniform numbers from 0 to 1 spread over more than 17 orders of magnitude and give 0.300080 for leading digit 1. Data spread evenly on the scale of digits (the logarithm) keep their distribution under any multiplication and follow $\log_{10}(1 + 1/d)$; uniform numbers within one order, or heights spread over only 0.170 of an order, do not** (Section 4). **Keywords**: Benford's law, leading digit, scale invariance, logarithmic uniformity, spread over orders of magnitude.

Scope of this paper (scope note): **No new theorem or law is claimed——that leading digits are uneven (Newcomb, 1881; Benford, 1938), that the only distribution unchanged by rescaling is this law (Pinkham, 1961), that numbers drawn from a mixture of distributions approach it (Hill, 1995), and that the fractional parts of $n \log_{10} 2$ spread evenly (Weyl, 1916) are all known.Applications are not treated——detecting fraud in accounts and the like are not entered.Why so much real data follows it is not discussed——data that follow and data that do not are only sorted and counted by how far they spread in orders of magnitude.Relation to earlier papers**: **Paper 117 separated four ways in which scale invariance fixes an exponent——this paper confirms in numbers that, for leading digits, a distribution unchanged by rescaling is $\log_{10}(1 + 1/d)$.**What is added is **counting the leading digits of 10000 powers of 2 exactly as integers**, **setting side by side that tripling changes nothing while doubling uniform numbers changes the shares**, and **placing the separator on whether the data spread across orders of magnitude, measuring the spread of uniform numbers, heights and products in orders**.

1. Notation and the Question

Setting: **let $D(x) \in \{1, \ldots, 9\}$ be the leading digit of a positive number x.Benford's law gives the share with leading digit d as**

$$P(D = d) = \log_{10}\left(1 + \frac{1}{d}\right),$$

which for $d = 1$ to 9 is $0.301030, 0.176091, 0.124939, 0.096910, 0.079181, 0.066947, 0.057992, 0.051153, 0.045757$, summing to 1.

Question: **which sequences of numbers follow these shares, when do they fail, and what separates the two?**

2. Powers of 2

Computation 2.1: **2^n was computed exactly as an integer for $n = 1$ to 10000, and the leading digits were counted.**

leading digit	count	share	$\log_{10}(1+1/d)$	difference
1	3010	0.301000	0.301030	−0.000030
2	1761	0.176100	0.176091	+0.000009
3	1249	0.124900	0.124939	−0.000039
4	970	0.097000	0.096910	+0.000090
5	791	0.079100	0.079181	−0.000081
6	670	0.067000	0.066947	+0.000053
7	579	0.057900	0.057992	−0.000092
8	512	0.051200	0.051153	+0.000047
9	458	0.045800	0.045757	+0.000043

The largest difference was **0.000092**. For the first 5000 Fibonacci numbers, the share with leading digit 1 was 0.301000 and the largest difference 0.000347.

> **Observation 2.2**
>
> 2^n has leading digit d when the fractional part of $n \log_{10} 2$ falls between $\log_{10} d$ and $\log_{10}(d+1)$.Since $\log_{10} 2$ cannot be written as a fraction, those fractional parts spread evenly over 0 to 1 as n grows——so the share with leading digit d is the width of the interval, $\log_{10}(d+1) - \log_{10} d = \log_{10}(1+1/d)$.

3. Changing the Scale

Computation 3.1: counting the leading digits of all 2^n multiplied by 3, the share with leading digit 1 was 0.301000 and the largest difference 0.000108.

> **Observation 3.2**
>
> multiplying by any number leaves the distribution unchanged.Tripling only adds $\log_{10} 3$ to the fractional part of $n \log_{10} 2$, which stays evenly spread——data whose leading-digit shares do not change when units change (yen to dollars, metres to feet) have no choice but to follow these shares.

4. The Separator

Computation 4.1: **for four sequences (200000 numbers each), the share with leading digit 1 and the number of orders of magnitude the data span were compared.**

data	share with leading digit 1	orders spanned
uniform numbers from 1 to below 10	0.110675	1
the same doubled	0.554595	1
heights (mean 170 cm, standard deviation 7 cm)	0.999985	0.170
products of 20 uniform numbers from 0 to 1	0.300080	17.44

The uniform 0.110675 is close to $1/9 = 0.111111$, and the doubled 0.554595 close to the $10/18 = 0.555556$ taken by 10 to 20. The largest difference between the products and $\log_{10}(1 + 1/d)$ was 0.001036.

> **Observation 4.2**
>
> **this is the core: the separator is whether the data spread across orders of magnitude.Uniform numbers and heights stay within one order, and where they gather in it decides the leading digit; doubling moves the gathering place and changes the shares——the products spread over more than 17 orders and become nearly even on the scale of digits. Then no multiplication changes the distribution, and the leading digits follow $\log_{10}(1 + 1/d)$.**

> **Observation 4.3**
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> **beside Paper 117.Paper 117 showed that scale invariance fixes an exponent in four different ways——for the distribution of leading digits, requiring invariance under rescaling fixes the shares uniquely as $\log_{10}(1 + 1/d)$. The powers of 2, unchanged by tripling, and the uniform numbers, changed by doubling, are its two sides.**

Observation 4.4

what is not said.Applications such as detecting fraud in accounts are not treated, and the theory of why so much real data follows the law is not entered——what is shown is that the shares fit for powers of 2, that rescaling does not change them, and that data not spanning orders of magnitude do not fit.

5. Roll Call and Closing

1. **leading digit 1 appears 3010 times among 2^n** (Computation 2.1): **0.301000.**
2. **all nine digits within 0.000092** (Computation 2.1): **with exact integers.**
3. **Fibonacci numbers within 0.000347** (Computation 2.1): **they follow too.**
4. **because the fractional parts spread evenly** (Observation 2.2): **interval widths become shares.**
5. **tripled, still 0.301000** (Computation 3.1): **unchanged by rescaling.**
6. **uniform numbers give 0.110675** (Computation 4.1): **close to $1/9$.**
7. **doubled, 0.554595** (Computation 4.1): **changed by rescaling.**
8. **heights give 0.999985** (Computation 4.1): **spanning only 0.170 of an order.**
9. **products of 20 give 0.300080** (Computation 4.1): **spanning more than 17 orders.**
10. **the separator is spreading across orders of magnitude** (Observation 4.2): **the core.**

Closing: **Counting leading digits in a list of numbers often gives about three in ten for 1.That holds only for data spread across many orders of magnitude——counting 10000 powers of 2 exactly, all nine leading digits were within 0.000092 of $\log_{10}(1 + 1/d)$, and tripling everything changed nothing, because the fractional parts of $n \log_{10} 2$ spread evenly.The separator is whether the data spread across orders of magnitude——uniform numbers from 1 to 10 are close to $1/9$, change to $10/18$ when doubled, and heights are almost all 1. Products of 20 uniform numbers spread over more than 17 orders and give 0.300080 for leading digit 1.Placed among the earlier papers——the scale invariance of Paper 117 fixes the shares uniquely for leading digits.To be honest——neither applications nor the theory of why real data follow the law are entered; only counts and comparisons of spread.**

References

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Yuuki YAMAGISHI (山岸 勇輝)

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