Freezing Time Grows in Proportion to Thickness for Thin Pieces and Toward Its Square for Thick Ones

— doubling the thickness multiplies the time by **2.044444** for a heat-transfer coefficient of **20** and a thickness of **1** cm, but by **3.639344** for **200** and **20** cm — which term matters is decided by the Biot number — [Paper 618]

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— *From thinking to syncing.* —

Abstract

When something watery is frozen in cold air, the time until it is fully frozen is set both by how fast heat passes from the surface to the air and by how fast it is conducted through the frozen layer (Plank's equation).How that time changes with thickness is obtained by two routes, formula and numerical computation.No new theorem or law is claimed. First, freezing time is the sum of a term proportional to the thickness and a term proportional to its square ——a slab cooled from both sides by air at −30 °C with a heat-transfer coefficient of 20 W/(m^2 K) freezes in 0.790720 hours at 1 cm, 4.305029 hours at 5 cm and 22.491582 hours at 20 cm. The interior term is the Biot number over 4 times the surface term (Section 2). **Second, and this is the core.The factor for doubling the thickness moves from 2 to 4 with the Biot number——for a coefficient of 20 and 1 cm (Biot number 0.090909) it is 2.044444, nearly proportional; for 200 and 20 cm (Biot number 18.181818) it is 3.639344, approaching the square. For thin pieces the surface is the narrow passage, for thick ones the interior** (Section 3). **Third, the numerical computation takes a little longer than Plank's equation——freezing the slab by the enthalpy method took 1.003827 times (Biot number 0.090909) to 1.058104 times (Biot number 18.181818) as long as Plank's equation. Doubling the grid moved it by only 0.04 per cent** (Section 4). **Fourth, the difference came from the heat of cooling the frozen part—— Plank's equation leaves out the heat needed to cool the frozen layer below 0 °C. Cutting the specific heat of the frozen part to one tenth in the computation shrank the difference from 1.015892 to 1.001589 and from 1.058104 to 1.005827, to about a tenth** (Section 4). **Fifth, shape matters too——at the same thickness (diameter) of 5 cm, a slab takes 4.305029 hours, a cylinder 2.152515 and a sphere 1.435010; the sphere freezes in a third of the slab's time** (Section 5). **Keywords**: freezing time, Plank's equation, Biot number, Stefan problem, food freezing.

Scope of this paper (scope note): **No new theorem or law is claimed——the equation for freezing time (Plank, 1913), the problem of a moving freezing front (Stefan, 1891), and methods for heat conduction (in the textbook by Carslaw and Jaeger and elsewhere) are all known.Food quality is not treated——the size of ice crystals and the relation between freezing and taste are not entered.Material values are representative——density 1000 kg/m^3, latent heat 334 kJ/kg, thermal conductivity of the frozen part 2.2 W/(m K), specific heat 2.1 kJ/(kg K), freezing temperature 0 °C.Only the case starting at the freezing temperature is considered——the time for pre-cooling is not included.Relation to earlier papers**: **Paper 465 showed that what sets how quickly things cool is not the specific heat but the ratio of surface area to volume, so that making something smaller shortens its cooling time constant in proportion to its size——the proportionality to thickness for thin pieces here is on that side; as pieces get thicker, conduction through the interior takes over and the dependence moves to the square.**What is added is **setting out, by Biot number, the factor for doubling the thickness**, **matching Plank's equation with an enthalpy-method computation and confirming in numbers the size and the cause of the difference**, and **placing the separator on the Biot number**.

1. Notation and the Question

Setting: **a slab of thickness a is cooled from both sides by air at temperature T_a. The material starts at its freezing temperature T_f, with density ρ, latent heat of freezing L, thermal conductivity of the frozen part k and surface heat-transfer coefficient h.Plank's equation gives the time to finish freezing as**

$$t = \frac{\rho L}{T_f - T_a}\left(\frac{P\,a}{h} + \frac{R\,a^2}{k}\right),$$

with $P = 1/2$ and $R = 1/8$ for a slab. Writing the Biot number as $\mathrm{Bi} = ha/k$, the second term is $\mathrm{Bi}/4$ times the first.

Question: **how does freezing time change with thickness, how accurate is Plank's equation, and what separates the cases?**

2. Freezing Time

Computation 2.1: **with air at -30 °C and representative material values, freezing times were found from Plank's equation.**

heat-transfer coefficient	thickness	Biot number	Plank
20 W/(m^2 K)	0.01 m	0.090909	**0.790720** h
20	0.05 m	0.454545	4.305029 h
20	0.20 m	1.818182	22.491582 h
200	0.01 m	0.909091	0.094886 h
200	0.05 m	4.545455	0.825863 h
200	0.20 m	18.181818	8.574916 h

3. Doubling the Thickness

Computation 3.1: **the factor by which freezing time grows when the thickness is doubled was found from Plank's equation and from the numerical computation (Section 4).**

coefficient	thickness	Plank	computation
20	0.01 → 0.02 m	2.044444	2.051491
20	0.05 → 0.10 m	2.204082	2.226169
20	0.20 → 0.40 m	2.625000	2.651287
200	0.01 → 0.02 m	2.370370	2.397872
200	0.05 → 0.10 m	3.063830	3.078759
200	0.20 → 0.40 m	**3.639344**	3.642009

> **Observation 3.2**
>
> **this is the core: the factor for doubling the thickness moves from 2 to 4 with the Biot number.With a small Biot number, heat crosses the interior easily and is held up where it leaves the surface for the air. The time is set by the surface term and is proportional to thickness——with a large Biot number, the surface cools at once, but heat is held up conducting through the frozen layer. The time is set by the interior term and approaches the square of the thickness.**

4. Numerical Computation

Computation 4.1: **independently of Plank's equation, half the slab was divided into 16 cells, and a computation following the enthalpy (including both latent heat and the heat of cooling the frozen part) found the time for every cell to finish freezing.**

coefficient, thickness	Plank	computation	ratio
20, 0.01 m	0.790720 h	0.793746 h	1.003827
20, 0.05 m	4.305029 h	4.373446 h	1.015892
20, 0.20 m	22.491582 h	23.345762 h	1.037978
200, 0.01 m	0.094886 h	0.097360 h	1.026073
200, 0.05 m	0.825863 h	0.867957 h	1.050970
200, 0.20 m	8.574916 h	9.073153 h	**1.058104**

Doubling to 32 cells (coefficient 20, 0.05 m) gave 4.375324 hours, a ratio of 1.000429 to 16 cells.

Computation 4.2: **cutting the specific heat of the frozen part to one tenth (210 J/(kg K)) in the computation shrank the ratio to Plank from 1.015892 to 1.001589 for coefficient 20 and 0.05 m, and from 1.058104 to 1.005827 for 200 and 0.20 m.**

> **Observation 4.3**
>
> **the computation takes longer than Plank because Plank leaves out the heat of cooling the frozen part.Plank's equation assumes the temperature falls linearly through the frozen layer from 0 °C toward the air, and does not count the heat needed to cool that layer——cutting the specific heat to a tenth shrank the difference to about a tenth, so the difference came from that heat. The larger the Biot number, the closer the frozen layer cools toward the air temperature, and the wider the difference.**

5. The Separator

> **Observation 5.1**
>
> **the separator is where the passage for heat is narrowest (the Biot number).If the surface is narrowest, time is proportional to thickness; if the interior, it approaches the square——with the same material and air, raising the coefficient from 20 to 200 moves the Biot number of a 20-cm slab from 1.818182 to 18.181818, and the doubling factor from 2.625000 to 3.639344.**

Computation 5.2: **changing the shape factors, at the same thickness (diameter) of 0.05 m and coefficient 20: slab 4.305029 h, cylinder 2.152515 h, sphere 1.435010 h.A sphere freezes in a third of the slab's time.**

> **Observation 5.3**
>
> **beside Paper 465.Paper 465 showed that how quickly things cool is set not by specific heat but by the ratio of surface area to volume, so making something smaller shortens its cooling time constant in proportion to its size——that is the small-Biot side, the same side as the proportionality to thickness for thin pieces here. As pieces thicken and the Biot number grows, freezing time moves to the square of the thickness, and making them smaller helps even more.**

> **Observation 5.4**
>
> **what is not said.Food quality is not treated; material values are representative, and pre-cooling time is not included——what is shown is that the doubling factor moves from 2 to 4 with the Biot number, and the size and cause of the difference between Plank's equation and the computation.**

6. Roll Call and Closing

1. **1 cm freezes in 0.790720 hours** (Computation 2.1): **coefficient 20, air at −30 °C.**
2. **the interior term is Bi/4 times the surface term** (Notation): **Plank's two terms.**
3. **doubling gives 2.044444 at Biot number 0.090909** (Computation 3.1): **nearly proportional.**
4. **doubling gives 3.639344 at Biot number 18.181818** (Computation 3.1): **approaching the square.**
5. **the factor moves from 2 to 4 with the Biot number** (Observation 3.2): **the core.**

6. **computation 1.003827 to 1.058104 times Plank** (Computation 4.1): **two routes.**
7. **doubling the grid moves it 0.04 per cent** (Computation 4.1): **the computation has converged.**
8. **a tenth of the specific heat gives a tenth of the difference** (Computation 4.2): **the cause confirmed.**
9. **the separator is where heat's passage is narrowest** (Observation 5.1): **the Biot number.**
10. **a sphere takes a third of a slab's time** (Computation 5.2): **shape factors.**

Closing: **The time to freeze something in cold air is the sum of a term for removing heat at the surface and a term for conducting it through the frozen layer.For thin pieces it is proportional to thickness; for thick pieces it approaches the square——doubling the thickness multiplies the time by 2.044444 at Biot number 0.090909 and by 3.639344 at 18.181818.An enthalpy-method computation took 0.4 to 5.8 per cent longer than Plank's equation——cutting the specific heat of the frozen part to a tenth shrank the difference to about a tenth, confirming that it comes from the heat of cooling the frozen layer, which Plank leaves out.The separator is where the passage for heat is narrowest, the Biot number——a narrow surface gives proportionality, a narrow interior the square; at the same thickness a sphere freezes in a third of a slab's time.Placed among the earlier papers——the "smaller cools faster in proportion to size" of Paper 465 is the small-Biot side; on the thick side it moves to the square.To be honest——material values are representative, and neither food quality nor pre-cooling is included.**

References

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