

$+C$ and Λ : A Duality on the Volume Degree of Freedom

— A Structural Interpretation with Unimodular Gravity as the Lens — [Paper 62]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

ORCID: 0009-0006-8587-3450

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ABSTRACT

In unimodular gravity (UG), the cosmological constant Λ appears not as a parameter of the action but as an **integration constant**, and is the quantity **canonically conjugate to the spacetime 4-volume** (Henneaux–Teitelboim 1989). Meanwhile, the volume restoration $\int 4\pi r^2 dr = (4/3)\pi r^3 + C$ (Paper 12) carries an integration constant $+C$ representing the **baseline (zero-point) of volume (mass)**. This paper judges their relation honestly, **as a structural duality rather than an identity**, via three tests: (i) dimensions --- $+C$ is L^3/M , Λ is $1/L^2$, which **differ**; (ii) the radial role in the Newtonian limit --- Λ acts as a uniform vacuum density ($\propto r^3$ in enclosed mass), $+C$ is a radius-independent baseline, a **different role**; (iii) yet since Λ is **conjugate to volume** and $+C$ is the **baseline of volume**, they are **two faces of the same “volume degree of freedom” (a duality)**. Conclusion: $+C \neq \Lambda$, but a genuine **structural resonance as a duality on the volume degree of freedom** holds. This duality is moreover **refined into the canonical pair (Λ, V_4)** : the conjugate of Λ is the spacetime 4-volume V_4 (HT), $+C$ is its **spatial 3-volume seed**, and the light cone $r = ct$ (Paper 61) **supplies the one dimension of time** assembling V_4 (unifying Papers 61, 62, and UG). This paper **does not solve the cosmological constant problem and does not predict Λ** (Paper 48).

1. Notation and Stance

Stance. This paper **claims no new physical law and does not solve the cosmological constant problem**. UG and “ $\Lambda =$ integration constant” are known. The contribution is to situate the $+C$ of volume restoration (Paper 12) relative to Λ , **separating identity, duality, and coincidence**. Stated at the outset: $+C$ and Λ are not identical, but dual on the volume degree of freedom. Here $4\pi = \int_{S^2} d\Omega$ (Paper 01).

2. Unimodular Gravity: Λ as Integration Constant, Conjugate to Volume (Known)

Proposition 1 (The two faces of Λ ; UG)

Fixing $\sqrt{-g} = \text{const}$ gives the trace-free Einstein equations $R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T)$. Divergence + Bianchi + $\nabla^\mu T_{\mu\nu} = 0$ gives $\partial_\mu(R + 8\pi GT) = 0$, i.e. $R + 8\pi GT = \text{const} = -4\Lambda$: Λ appears as an **integration constant**. In the covariant formulation (Henneaux–Teitelboim 1989), Λ is the quantity **canonically conjugate to the spacetime 4-volume** $\int \sqrt{-g} d^4x$.

Thus in UG, Λ is not a fixed parameter of the action but an **integration constant and a degree of freedom conjugate to volume**.

3. Volume Restoration and Its Baseline $+C$

Proposition 2 (The integration constant of volume restoration)

$\int 4\pi r^2 dr = \frac{4}{3}\pi r^3 + C$ (Paper 12); $+C$ is the integration constant representing the **baseline (zero-point) of volume (mass)**. For $\int_0^r$ it cancels between the limits; for the indefinite integral it always appears. Its dimension is L^3 (or M).

4. Testing the Bridge: Identity or Duality

Proposition 4 (Exclusion by dimension)

$+C$ has dimension L^3/M ; Λ has dimension $1/L^2$. Since these differ, the **literal equality $+C = \Lambda$ does not hold**.

Proposition 5 (Different radial roles in the Newtonian limit)

In $\nabla^2\Phi = 4\pi G\rho - \Lambda c^2$, the Λ term acts as a **uniform vacuum density** ($\propto r^3$ in enclosed mass); $+C$ is a **radius-independent baseline** (a central point mass). Hence **different roles** (Λ : r^3 ; $+C$: constant).

Proposition 6 (Duality on the volume degree of freedom)

By Props 4–5, $+C \neq \Lambda$. But by Prop 1, Λ is **conjugate to volume**, and by Prop 3, $+C$ is the **baseline of volume**. Hence they are **two faces of the same “volume degree of freedom”**:

$+C$	the baseline (zero-point) of volume
Λ	the variable conjugate to volume (UG)

A **duality on the volume degree of freedom**: not identity, but two faces of the same root.

5. Verdict and Discipline

Corollary 1 (Verdict)

$+C$ and Λ are **not identical** (dimensions and radial roles differ), but a structural resonance holds as a **duality on the volume degree of freedom** ($+C$ = baseline, Λ = conjugate). This is not identification by numerical coincidence but a **structural duality** (Paper 48 upheld).

Honest boundaries:

- This paper **does not solve the cosmological constant problem**. UG recasts Λ as an integration constant but **does not predict its value** ($\sim 10^{-120}$), and whether UG helps is **debated**.
- $+C \neq \Lambda$. What is shown is a **duality (structural resonance)**, not an identity.
- The intuition “ $+C$ = the cosmic baseline” meshes, **as a duality**, with the genuine UG structure “ Λ = conjugate to volume” --- the contribution.

6. Canonical Refinement: the Pair (Λ, V_4) and the Light Cone (Unifying Paper 61)

Examined precisely, in the canonical pair of Proposition 1, Λ is conjugate to the **spacetime 4-volume** $V_4 = \int \sqrt{-g} d^4x$: $\{\Lambda/8\pi G, V_4\} \sim 1$.

Proposition 7 ($(+C, \Lambda)$ is not a canonical pair)

The canonical conjugate of Λ is V_4 (dimension L^4). But $+C$ is the baseline of a spatial **3**-volume (L^3), not V_4 ($L^3 \neq L^4$). Hence $(+C, \Lambda)$ is **not** a canonical conjugate pair; the duality of §5 is not promoted to canonical conjugacy.

Proposition 8 (The light cone supplies time; $+C$ is the seed of V_4)

The gap between L^3 and L^4 is the light-cone time $c dt$. By the light sphere $r = ct$ (Paper 61), $V_4 = \int(\text{spatial 3-volume}) c dt$, whose spatial **3**-volume has baseline $+C$. Therefore

$$+C \text{ (3-vol seed; this paper)} \xrightarrow{\int c dt \text{ (light cone; Paper 61)}} V_4 \text{ (4-vol)} \longleftrightarrow \Lambda \text{ (conjugate; HT)}.$$

The canonical pair is (Λ, V_4) ; $+C$ is its **3**-volume seed, and Paper 61's light cone supplies the one dimension of time. **Papers 61, 62, and UG (HT) close into one canonical structure.**

Corollary 2 (Refinement of the duality)

The “duality on the volume degree of freedom” of §5 is refined into the canonical pair (Λ, V_4) with $+C$ (the **3**-volume seed) and the light cone (Paper 61, supplying time). $(+C, \Lambda)$ itself is not a conjugate pair ($L^3 \neq L^4$). The cosmological constant problem remains unsolved.

7. Conclusion

1. In UG, Λ is an integration constant conjugate to the spacetime **4**-volume (known; Prop 1).
2. The $+C$ of $\int 4\pi r^2 dr = (4/3)\pi r^3 + C$ is the baseline of volume (mass) (Prop 2).
3. $+C$ (L^3/M) and Λ ($1/L^2$) differ in dimension and radial role (Λ : r^3 ; $+C$: constant); hence **not identical** (Props 4–5).
4. But since Λ is conjugate to volume and $+C$ is the baseline of volume, they form a **duality on the volume degree of freedom** (Prop 6, Cor 1).
5. This duality is refined into the **canonical pair** (Λ, V_4) : the conjugate of Λ is the spacetime **4**-volume V_4 , $+C$ is its spatial **3**-volume seed, and the light cone $r = ct$ (Paper 61) supplies the one dimension of time assembling V_4 . $(+C, \Lambda)$ itself is not a conjugate pair ($L^3 \neq L^4$). Papers 61, 62, and UG close into one canonical structure (Props 7–8, Cor 2).

This paper claims no new physical law and does not solve the cosmological constant problem. The integration constant $+C$ of volume restoration corresponds structurally to Λ of UG (an integration constant conjugate to volume) --- **not as an identity but as a duality on the volume degree of freedom.** The intuition “ $+C = \text{the cosmic baseline}$ ” is thereby placed precisely, as this duality. Identity and duality (resonance) are kept distinct.

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Yuuki YAMAGISHI (山岸 勇輝)

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