

The Collatz Map Shrinks on Average, but Replacing 3 by 5 Makes It Grow

— Triple an odd number and add 1, halve an even one — replacing 3 by 5 gives $\sqrt{5/4} = 1.118034$, growing on average, and **9274** of the numbers from 1 to 10000 exceeded 10^{12} — [Paper 621]

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— *From thinking to syncing.* —

Abstract

Repeatedly replacing a positive integer n by $3n + 1$ if it is odd and by $n/2$ if it is even is conjectured to reach 1 from every starting number (the Collatz conjecture). Its behaviour is checked in numbers, to see why it shrinks and what change makes it stop shrinking. No new theorem or law is claimed, and the conjecture is not proved. First, every number up to 10^6 reaches 1: running every number from 2 to 10^6 , each eventually fell below its starting value and reached 1. The mean number of steps is **131.434555** and the longest is 837799 with 524 steps. 27 takes 111 steps and climbs to 9232 on the way (Section 2). Second, on average each step shrinks: writing an odd step as $(3n + 1)/2$, a step multiplies by about 1.5 half the time and by 0.5 half the time, with geometric mean $\sqrt{3/4} = \mathbf{0.866025}$. Running 2000 random large numbers for 40 steps each gave a measured **0.864706** (Section 3). Third, and this is the core. The separator is whether the multiplier is less than 4: replacing 3 by 5 gives a geometric mean of $\sqrt{5/4} = \mathbf{1.118034}$, growing on average. Of the numbers from 1 to 10000, 9274 exceeded 10^{12} and only 726 fell into cycles (Section 4). Fourth, shrinking on average does not guarantee reaching 1: replacing $3n + 1$ by $3n - 1$ keeps the same geometric mean $\sqrt{3/4}$ and shrinks on average, yet the numbers from 1 to 10000 split into three cycles: the one containing 1 (3244), the one containing 5 (3213) and the cycle of length 18 containing 17 (3543). Shrinking does not decide which cycle is reached (Section 4). **Keywords:** Collatz conjecture, $3n+1$ problem, geometric mean, stopping time, dynamical system.

Scope of this paper (scope note): No new theorem or law is claimed: the $3x + 1$ problem and its generalisations (Lagarias, 1985), properties of stopping times (Terras, 1976) and probabilistic heuristics are all known (collected in the book edited by Lagarias). The conjecture is not proved: checking up to 10^6 is not a proof. “Shrinking on average” is a heuristic assuming odd and even appear equally often and guarantees nothing about any particular number. The general theory of $an + b$ is not treated: only $3n + 1$, $3n - 1$ and $5n + 1$ are compared. Relation to earlier papers: Paper 195 showed that “stable” splits into six separate words by what is perturbed: the cycle $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$ here attracts by shrinking on average, and with the same shrinking $3n - 1$ has three attracting cycles. What is added is counting the steps up to 10^6 in a table and matching a sample against running each number through, obtaining the geometric mean of the factor by formula and by random numbers, and placing the separator on the multiplier versus 4 and checking it with $5n + 1$ and $3n - 1$.

1. Notation and the Question

Setting: map a positive integer n to $an + 1$ if odd and to $n/2$ if even ($a = 3$ is the Collatz map). The number of applications until 1 is reached is called the number of steps.

Question: does every number reach 1, why does it shrink, and what happens when a changes?

2. Running Up to 10^6

Computation 2.1: each number from 2 to 10^6 was run until it fell below its starting value, and the remaining steps were added from the table. Every number eventually fell below its start, so all reached 1.

quantity	value
mean number of steps ($2\text{--}10^6$)	131.434555
longest	837799, 524 steps
steps and peak for 27	111 steps, 9232

The table agreed with running each number through to the end for 2000 randomly chosen numbers.

Observation 2.2

even small numbers take long detours. 27 climbs to 9232 before returning to 1 in 111 steps: the rises and falls are wild, but not one number up to 10^6 avoided 1.

3. How It Shrinks on Average

Observation 3.1

combine an odd step with the halving that always follows. Taking odd n to $(3n + 1)/2$ and even n to $n/2$, a step multiplies by roughly 1.5 or 0.5. If odd and even appear equally often, the geometric mean of the factor is

$$\sqrt{\frac{3}{2} \times \frac{1}{2}} = \sqrt{\frac{3}{4}} = 0.866025,$$

shrinking on average.

Computation 3.2: choosing 2000 random numbers between 10^{20} and 10^{21} and running each 40 steps, the geometric mean of the factor per step was measured as **0.864706**.

Observation 3.3

in checking, I once measured with a bias. At first every starting number was forced to be odd, so the first step always grew, and the measurement was 0.878061. It matched the prediction 0.878000 of the heuristic with only the first step fixed at 1.5, so after confirming the source of the bias the starting numbers were made random.

4. The Separator

Computation 4.1: changing a , the destinations of the numbers from 1 to 10000 were counted (exceeding 10^{12} counted as “escaped”).

map	geometric mean of factor	destinations of 1–10000
$3n + 1$	$\sqrt{3/4} = 0.866025$	all in the cycle containing 1
$3n - 1$	$\sqrt{3/4} = 0.866025$	cycle of 1: 3244 ; of 5: 3213 ; of 17: 3543
$5n + 1$	$\sqrt{5/4} = 1.118034$	escaped 9274 ; cycles 726

The 726 numbers under $5n + 1$ that fell into cycles went to the cycle containing 13 (383), the cycle of length 7 containing 1 (256) and the cycle containing 17 (87).

Observation 4.2

this is the core: the separator is whether the multiplier a is less than 4. The geometric mean of the factor is $\sqrt{a/4}$: below 4 it shrinks on average, above 4 it grows: with $a = 5$ more than nine tenths of the numbers from 1 to 10000 exceeded 10^{12} . The difference between 3 and 5 is which side of 4 they are on.

Observation 4.3

shrinking on average does not guarantee reaching 1. $3n - 1$ shrinks exactly as $3n + 1$ does, yet the numbers from 1 to 10000 split into three cycles: shrinking only gives the heuristic that numbers do not grow without limit; how many cycles there are to fall into is a separate question. That only the cycle containing 1 has been found for $3n + 1$ does not follow from the way it shrinks.

Observation 4.4

beside Paper 195. Paper 195 showed that “stable” splits into six by what is perturbed: the Collatz cycle attracts in the sense of shrinking on average. But $3n - 1$, which shrinks the same way, has three attracting cycles. Attracting and being the only one are different.

Observation 4.5

what is not said. The conjecture is not proved; checking up to 10^6 says nothing beyond. That numbers under $5n + 1$ really grow without limit is not proved either. What is shown is the count up to 10^6 , the formula and measurement for the average shrinking, and the difference when the multiplier changes.

5. Roll Call and Closing

1. every number up to 10^6 reaches 1 (Computation 2.1): **table and direct runs agree.**
2. mean steps **131.434555**; longest **837799 with 524** (Computation 2.1): **agreeing with known values.**
3. **27** climbs to **9232** and takes **111** steps (Observation 2.2): **a wild detour.**
4. geometric mean $\sqrt{3/4} = \mathbf{0.866025}$ (Observation 3.1): **shrinking on average.**
5. measured **0.864706** (Computation 3.2): **two routes.**
6. forcing odd starts biases it to **0.878** (Observation 3.3): **confirmed by the prediction 0.878000.**
7. $5n + 1$ gives $\sqrt{5/4} = \mathbf{1.118034}$ (Computation 4.1): **9274 escape.**
8. the separator is the multiplier versus **4** (Observation 4.2): **the core.**
9. $3n - 1$ shrinks the same way into three cycles (Observation 4.3): **shrinking does not fix the destination.**
10. attracting is not the same as being the only one (Observation 4.4): **Paper 195.**

Closing: Tripling odd numbers and adding 1, halving even ones, every number up to 10^6 reached 1. Because on average each step shrinks: combining an odd step, a step multiplies by 1.5 or 0.5, with geometric mean $\sqrt{3/4} = 0.866025$, measured as 0.864706 with random numbers. The separator is whether the multiplier is less than 4: replacing 3 by 5 gives $\sqrt{5/4} = 1.118034$, growing on average, and 9274 of the numbers from 1 to 10000 exceeded 10^{12} . And shrinking does not guarantee reaching 1: $3n - 1$ shrinks the same way and split into three cycles. Placed among the earlier papers. It is a cycle attracting in one of the senses of “stable” in Paper 195, but whether the attracting cycle is the only one is a separate question. To be honest: the conjecture is not proved; the numbers up to 10^6 were counted and the average heuristic measured.

References

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