On a Board with an Odd Number of Squares, No Knight's Tour Returns to Its Start

— searching the whole 5×5 board, tours visiting every square once number 1728 counting direction, and those returning to the first square 0 — 6×6 and 8×8 do have returning tours — [Paper 625]

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— *From thinking to syncing.* —

Abstract

A chess knight jumps 2 squares one way and 1 the other. A path visiting every square of the board exactly once is a knight's tour, and one whose last square is a knight's move from the first is a closed tour.Which boards have closed tours is checked by two routes, colouring and exhaustive search.No new theorem or law is claimed. First, 5×5 has no closed tour——searching every path on the 5×5 board, tours visiting every square once number 1728 counting direction, and 0 of them return to the first square (Section 2). **Second, tours start only on the larger colour——colouring the board as a chequerboard, 5×5 has 13 squares of one colour (including the four corners) and 12 of the other. All 1728 tours start on the 13-square colour and not one on the 12-square colour. Tours starting at a corner number 304** (Section 2). **Third, and this is the core.Since a knight changes colour at every move, a board with an odd number of squares has no closed tour——stepping on the colours alternately, a path over 25 squares starts and ends on the larger colour. The move from the last square back to the first would join two squares of the same colour, which a knight cannot do** (Section 3). **Fourth, boards with an even number of squares do have closed tours——on 6×6 and 8×8, a search trying the squares with fewest onward moves first found closed tours, confirmed to visit every square once by knight's moves and return to the start** (Section 4). **Keywords**: knight's tour, closed tour, two-colouring, Warnsdorff's rule, Hamiltonian cycle.

Scope of this paper (scope note): **No new theorem or law is claimed——the classification of rectangular boards with closed tours (Schwenk, 1991), the rule of moving to the square with fewest onward moves (Warnsdorff, 1823), and the number of tours on 5×5 are known.General formulas for the number of tours are not treated——5×5 is searched fully, and on 6×6 and 8×8 one closed tour each is found.Even boards without closed tours are not treated——boards such as $4 \times n$ are known to have no closed tour despite an even number of squares; only the odd case is examined here.Relation to earlier papers**: **Paper 601 showed that half the positions of the fifteen puzzle cannot be reached because the board can be coloured in two colours——the same two-colouring here forbids closed tours.Paper 348 showed that on the same graph connecting is cheap and ordering is not——an ordering visiting every square once is costly to search for.**What is added is **counting the tours and the colours they start on by an exhaustive search of 5×5, matching the colour count and the exhaustive search**, and **placing the separator on the parity of the number of squares**.

1. Notation and the Question

Setting: **colour the squares of an $R \times C$ board as a chequerboard. A knight's move goes $(2, 1)$ or $(1, 2)$ and always lands on the other colour.A path visiting every square once is a tour; a tour whose last square is a knight's move from its first is a closed tour.**

Question: **which boards have closed tours, and from which squares can tours start?**

2. Searching All of 5×5

Computation 2.1: **from each of the 25 squares of 5×5, every path was searched.**

counted	number
tours, counting direction	**1728**
closed tours	**0**
tours starting on the larger colour (**13** squares, incl. corners)	**1728**
tours starting on the smaller colour (**12** squares)	**0**
tours starting at a corner	**304**

Observation 2.2

not one of the 1728 tours returns to its start.And not one tour starts on the 12 squares of the smaller colour——on the same board with the same piece, the possible starting squares divide by colour alone.

3. Counting by Colour

Observation 3.1

this is the core: since a knight changes colour at every move, a board with an odd number of squares has no closed tour.A tour steps on the colours alternately, so a path over 25 squares visits colours A and B as A, B, A, B, …, A, 13 to 12. It therefore starts and ends on the larger colour——the last and first squares have the same colour, and a knight cannot join them in one move. Starting on the smaller colour, the count cannot come out 13 to 12 and the path cannot cover every square.

Observation 3.2

the colour count predicts the exhaustive search exactly.0 closed tours and 0 tours from the smaller colour both follow from the colour counts without examining a single path——the exhaustive search confirmed that prediction over all 1728 tours.

4. Even Boards

Computation 4.1: **on 6×6 and 8×8, closed tours were sought by a search that tries squares with fewest onward moves first and prunes when some unvisited square can no longer keep two edges.One was found after 158204 branch points on 6×6 and after 67 on 8×8, each confirmed to visit every square once by knight's moves and return to the start.**

Observation 4.2

on even boards the two colours are equal in number, and closed tours become possible.36 squares give 18 to 18, 64 give 32 to 32, and the constraint of starting and ending on the larger colour disappears——but an even number does not guarantee a closed tour: $4 \times n$ boards are known to have none.

Observation 4.3

in checking, I once searched the wrong way.Merely repeating the fewest-onward-moves rule with random tie-breaking found no closed tour on 6×6 in 300 attempts——that rule finds open tours well but pays no attention to returning to the start. Switching to a pruned search found one.

5. The Separator

Observation 5.1

the separator is the parity of the number of squares (whether the two colours are equal).With an odd number, the colours differ by one, there is no closed tour, and tours start only on the larger colour——with an even number, the colours are equal and that constraint disappears.

Observation 5.2

beside Papers 601 and 348.Paper 601 showed that half the positions of the fifteen puzzle cannot be reached because the board is two-colourable; Paper 348 that an ordering visiting everything once costs more than merely connecting——here the same two-colouring forbids closed tours and predicts the answer without paying the high cost of an exhaustive search.

Observation 5.3

what is not said.General formulas for the number of tours and the classification of even boards without closed tours are not treated——what is shown is the agreement of the exhaustive search of 5×5 with the colour count, and closed tours found on even boards.

6. Roll Call and Closing

1. **1728 tours on 5×5** (Computation 2.1): **counting direction.**
2. **0 closed tours** (Computation 2.1): **none returns.**
3. **0 tours from the smaller colour** (Computation 2.1): **only from the larger.**

4. **304 from a corner** (Computation 2.1): **corners are the larger colour.**
5. **a knight changes colour every move** (Observation 3.1): **the core.**
6. **13 to 12: starting and ending on the larger colour** (Observation 3.1): **no move back.**
7. **the colour count predicts the exhaustive search** (Observation 3.2): **two routes.**
8. **closed tours on 6×6 and 8×8** (Computation 4.1): **found and confirmed.**
9. **the random rule found no closed 6×6 tour** (Observation 4.3): **the search was changed.**
10. **the separator is the parity of the number of squares** (Observation 5.1): **the same colouring as Paper 601.**

Closing: **Can a knight visit every square of a board once and return to the first square?On a board with an odd number of squares, it cannot——searching all of 5×5, 0 of the 1728 tours returned to the start, and every tour began on the 13-square colour, none on the 12-square colour.Because a knight changes colour at every move——a path over 25 squares starts and ends on the larger colour, and those two cannot be joined in one move. The colouring predicted the exhaustive search without examining a single path.The separator is the parity of the number of squares——on 6×6 and 8×8 the colours are equal, and closed tours were found.Placed among the earlier papers——the two-colouring of Paper 601 here forbids closed tours and answers in advance the costly ordering of Paper 348.To be honest——neither general formulas for the number of tours nor even boards without tours are entered.**

References

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