A Game with Infinite Expectation Has an Expectation of About 41 Once the Bank Is Limited to 10^{12}

— if the bank can pay at most B, the expectation exceeds $\log_2 B$ by only 0.947435 to 0.987341 — measured by logarithmic utility the certain equivalent is 4, and the average over n plays without a cap keeps growing with n and never settles — [Paper 629]

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— From thinking to syncing. —

Abstract

A coin is tossed until heads, and 2^k is paid if the first head comes on toss k (the St. Petersburg game).Its expectation is infinite. What becomes of that infinity when something is made finite is counted.No new theorem or law is claimed. First, the expectation is infinite——the first head comes on toss k with probability $1/2^k$ and pays 2^k, so every k adds exactly 1 to the expectation. The terms never run out, so the sum grows without bound (Section 2). **Second, limiting the bank to B brings the expectation to about $\log_2 B$——if only B can be paid once 2^k exceeds B, the expectation is $K + B/2^K$ (with K the largest integer for which 2^K does not exceed B), and even with $B = 10^{12}$ it is 40.818989. Up to toss 39 each adds 1, and everything beyond adds 1.818989 together. Each thousandfold increase of the bank adds only 9.954224 to 9.957367** (Section 3). **Third, measured by logarithmic utility, the certain equivalent is 4——measuring not the amount but the expectation of its natural logarithm (Bernoulli), the value is $2 \ln 2 = 1.386294$, and the certain amount of equal worth is exactly 4** (Section 4). **Fourth, and this is the core.Making only the number of plays finite, without a cap, the average does not settle——the average prize over n plays was simulated and the median of 41 repetitions taken: 8.62 for $n = 100$ and 22.295930 for $n = 10^6$. Multiplying n by 10^4 raised the average by 13.675930, differing by 0.388218 from $\log_2 10^4 = 13.287712$. The average keeps growing like the logarithm of n** (Section 5). **Fifth, the separator is what is made finite——a finite bank brings the expectation to about $\log_2 B$, logarithmic utility gives a certain equivalent of 4, and a finite number of plays gives a finite average that keeps growing with n. The infinity remains in a different form depending on where it is cut** (Section 6). **Keywords**: St Petersburg paradox, expected value, logarithmic utility, bankroll limit, Daniel Bernoulli.

Scope of this paper (scope note): **No new theorem or law is claimed——that this game has infinite expectation, that logarithmic utility makes it finite (Daniel Bernoulli, 1738), and that the average over plays grows like the logarithm of n (Feller, 1945) are all known.Gambling is not recommended.The merits of utility theory are not discussed——logarithmic utility is only computed as one way of making the game finite.The simulation uses one random seed——the deviation of the median from $\log_2 n$ ($+1.976144$ to $+3.312216$) moves with the seed.Relation to earlier papers**: **Paper 298 showed that a doubling strategy winning 99.9999% of the time still has expectation exactly 0, because of the fence of capital——here the same fence of capital shrinks an infinite expectation to about $\log_2 B$.**What is added is **matching the capped expectation by two routes (the sum by definition and $K + B/2^K$), a simulation varying the number of plays from 10^2 to 10^6**, and **placing the separator on what is made finite**.

1. Notation and the Question

The game: **a coin is tossed until heads. If the first head comes on toss k, 2^k is paid. The probability that the first head comes on toss k is $1/2^k$.**

Question: **when the expectation is infinite, what remains if the bank, the utility or the number of plays is made finite?**

2. The Expectation Is Infinite

Observation 2.1

the term for toss k is the product of the probability $1/2^k$ and the amount 2^k, exactly 1.Every k adds 1 and the terms never run out, so the expectation grows without bound.Large amounts are rare, but each halving of the probability comes with a doubling of the amount, so what is added does not shrink.

3. Limiting the Bank

Computation 3.1: **with only B paid once 2^k exceeds B, the expectation was found by two routes, the sum by definition $\sum_k \min(2^k, B)/2^k$ and the formula $K + B/2^K$ (with K the largest integer for which $2^K \le B$), and they agreed.**

bank B	expectation	$\log_2 B$	difference	$B/2^K$
10^3	10.953125	9.965784	0.987341	1.953125
10^6	20.907349	19.931569	0.975780	1.907349
10^9	30.862645	29.897353	0.965292	1.862645
10^{12}	**40.818989**	39.863137	0.955852	1.818989
10^{15}	50.776357	49.828921	0.947435	1.776357

> **Observation 3.2**
>
> up to toss K each adds 1, and everything beyond is cut to B, adding $B/2^K$ together (at least 1 and less than 2).For 10^{12} this is 39 from the first 39 tosses and 1.818989 from the rest——each thousandfold increase of the bank adds only 9.954224 to 9.957367, close to $\log_2 1000 = 9.965784$. The infinite expectation shrinks to the logarithm of the bank.

4. Logarithmic Utility

Computation 4.1: the expectation of the natural logarithm of the amount is $\sum_k k \ln 2/2^k = 2 \ln 2 = 1.386294$.The certain amount of equal worth is $e^{2 \ln 2} = 4.000000$.

> **Observation 4.2**
>
> measured by the logarithm, the term for toss k becomes $k \ln 2/2^k$, and the halving of probability outpaces the doubling of the amount.So the sum is finite——what was made finite is not the amount paid but the way it is measured.

5. Limiting the Number of Plays

Computation 5.1: the average prize over n plays without a cap was simulated, repeated 41 times, and the median taken.

n	median of the average	$\log_2 n$	difference
10^2	8.620000	6.643856	+1.976144
10^3	13.278000	9.965784	+3.312216
10^4	15.764000	13.287712	+2.476288
10^5	19.714700	16.609640	+3.105060
10^6	**22.295930**	19.931569	+2.364361

From $n = 10^2$ to 10^6 the median of the average rose by 13.675930, differing by 0.388218 from $\log_2 10^4 = 13.287712$.

> **Observation 5.2**
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> this is the core: making the number of plays finite makes the average finite, but it keeps growing as the number increases, with no value to settle at.The more plays, the more likely a rare large amount is met, so the average grows like the logarithm of n——reading “over many plays the average settles at the expectation” belongs to the case where the expectation is finite.

6. The Separator

> **Observation 6.1**
>
> the separator is what is made finite.With the bank, the expectation is about $\log_2 B$; with the utility, the certain equivalent is 4; with the number of plays, the average keeps growing like $\log_2 n$——the same infinity remains, depending on where it is cut, as the logarithm of the bank, a single constant, or the logarithm of the number of plays.

> **Observation 6.2**
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> beside Paper 298.In Paper 298, a doubling strategy winning 99.9999% of the time had expectation exactly 0 because of the fence of capital——here the same fence of capital shrinks an infinite expectation to about $\log_2 B$. In both, capital rather than probability decides the answer.

Observation 6.3

what is not said.What one should pay for this game is not stated, the merits of utility theory are not discussed, and gambling is not recommended——what is shown is the number that remains under each of three ways of making the game finite.

7. Roll Call and Closing

1. **every k adds 1** (Observation 2.1): **the expectation is infinite.**
2. **40.818989 with a bank of 10^{12}** (Computation 3.1): **two routes agree.**
3. **0.947435 to 0.987341 above $\log_2 B$** (Computation 3.1): **shrunk to the logarithm of the bank.**
4. **about 9.95 more per thousandfold** (Observation 3.2): **close to $\log_2 1000$.**
5. **4 by logarithmic utility** (Computation 4.1): **the exponential of $2 \ln 2$.**
6. **the measure was made finite** (Observation 4.2): **not the amount paid.**
7. **22.295930 for $n = 10^6$** (Computation 5.1): **median of the average.**
8. **13.675930 more for 10^4 times the plays** (Computation 5.1): **0.388218 from $\log_2 10^4$.**
9. **the average does not settle** (Observation 5.2): **the core.**
10. **the fence of capital decides** (Observation 6.2): **Paper 298.**

Closing: **The game of tossing until heads and receiving 2^k has infinite expectation, since every k adds 1.Limiting the bank to B leaves an expectation less than 1 above $\log_2 B$: 40.818989 even with 10^{12}.Measured by logarithmic utility the certain equivalent is 4.Limiting only the number of plays makes the average finite, but it keeps growing with the number and does not settle——multiplying n by 10^4 raised the median of the average by 13.675930.The separator is what is made finite.Placed among the earlier papers——the fence of capital that made the expectation 0 in Paper 298 here shrinks the infinity to the logarithm of the bank.To be honest——neither what one should pay for this game nor the merits of utility theory are entered.**

References

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