For a Circular Orbit More Than 11.94 Times Farther Out, Going Even Farther First and Coming Back Needs Less Fuel

— once the target radius exceeds 11.938765 times the start, a bi-elliptic transfer with B far enough needs a smaller total change of velocity — beyond 15.581719 times, the bi-elliptic transfer is cheaper for every B — but at 20 times the radius with $B = 100$ it takes 24 times as long — [Paper 637]

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September 2026

— From thinking to syncing. —

Abstract

A craft moves from a circular orbit of radius 1 to one of radius R. The Hohmann transfer goes in two burns, one onto an ellipse and one to circularise. The bi-elliptic transfer goes in three: onto an ellipse with apoapsis B ($B > R$), a burn at B raising the periapsis to R, and a burn at R to circularise.Since more fuel is needed the larger the total change of velocity Δv, the transfers are compared by total Δv (with the speed on the starting circle as 1).No new theorem or law is claimed. First, there are two boundaries——the bi-elliptic transfer with B taken ever farther equals the Hohmann transfer at $R = 11.938765$. The Hohmann Δv reaches its maximum of 0.536258 at $R = 15.581719$, and for targets beyond that the bi-elliptic transfer was cheaper for every B. At $R = 13$, between the two boundaries, it is cheaper only when B lies beyond 48.904843 (Sections 2 and 3). **Second, and this is the core.The Hohmann Δv does not keep rising as the target gets farther——beyond $R = 3.304167$, the Δv to enter the target circular orbit exceeds the Δv to escape to infinity from the start ($\sqrt{2} - 1 = 0.414214$). Entering a far circular orbit costs more than escaping outright. So a path going out almost far enough to escape, then coming back, can be cheaper** (Section 2). **Third, set side by side, the bi-elliptic transfer burns more at the start and a little far out——at $R = 20$ and $B = 100$, the Hohmann transfer takes $0.380131 + 0.154600 = 0.534731$ and the bi-elliptic $0.407195 + 0.043663 + 0.065068 = 0.515927$, 0.018805 less. But it takes 24.207343 times as long** (Section 4). **Fourth, the formula's Δv really arrives——integrating the orbits numerically, both transfers, with the formula's Δv, passed through their apoapses and periapses and reached the circular orbit of radius 20 within 10^{-5}. From a 300 km circular Earth orbit to a circular orbit at the Moon's distance ($R = 57.560963$), the Hohmann transfer needs 3.936544 km/s against 3.621910 km/s for the bi-elliptic transfer going far out** (Sections 5 and 6). **Keywords**: Hohmann transfer, bi-elliptic transfer, orbital transfer, delta-v, orbital mechanics.

Scope of this paper (scope note): **No new theorem or law is claimed——the Hohmann transfer (Hohmann, 1925), the bi-elliptic transfer (Sternfeld, 1934), and the two boundaries 11.94 and 15.58 are known.Real launch planning is not discussed——time constraints, the gravity of other bodies, and the duration of burns are not included; for time only the ratio is given.Why the bi-elliptic transfer gains is not pinned to a single reason——only the breakdown of burns is set out.Relation to earlier papers**: **Paper 506 showed that the same fuel yields more energy the faster the craft is moving when it burns——here the total Δv of two burns straight to the target is compared with three burns going the long way, and the boundaries where the long way becomes cheaper are counted. The bi-elliptic transfer burns more than the Hohmann at the start, where the speed is greatest, and a little at the far apoapsis, where the speed is small.**What is added is **finding the boundaries 11.938765 and 15.581719 and the 3.304167 beyond which the Hohmann transfer costs more than escape**, **finding the B at which the bi-elliptic transfer becomes cheaper for $R = 13$**, **checking the formula's Δv by numerical integration of the orbits**, and **placing the separator on the ratio of radii R**.

1. Notation and the Question

Units: **the radius of the starting circular orbit is 1, and the speed on it is 1.The radius of the target circular orbit is R, and the intermediate apoapsis of the bi-elliptic transfer is B.**

The Hohmann transfer: **it rides an ellipse with periapsis 1 and apoapsis R. The total change of velocity is**

$$H(R) = \left(\sqrt{\frac{2R}{1+R}} - 1\right) + \left(\sqrt{\frac{1}{R}} - \sqrt{\frac{2}{R(1+R)}}\right).$$

The bi-elliptic transfer: **it rides an ellipse with periapsis 1 and apoapsis B, raises the periapsis to R at B, and circularises at R. As B is taken ever farther, the total approaches**

$$H_\infty(R) = (\sqrt{2} - 1)\left(1 + \frac{1}{\sqrt{R}}\right).$$

Question: **from which R does the long way become cheaper?**

2. Two Boundaries

Computation 2.1: **$H(R) = H_\infty(R)$ at $R = 11.938765$. $H(R)$ reaches its maximum of 0.536258 at $R = 15.581719$. $H(R)$ begins to exceed the Δv to escape to infinity from the start, $\sqrt{2} - 1 = 0.414214$, at $R = 3.304167$.**

Observation 2.2

this is the core: the Hohmann Δv does not keep rising as the target gets farther.Beyond $R = 3.304167$, entering the far circular orbit needs more Δv than escaping to infinity outright. Past $R = 15.581719$, the Hohmann Δv begins to fall, approaching $\sqrt{2} - 1$ as R grows without bound——this is why a path going out almost far enough to escape, then coming back with a small burn, can be cheaper than going straight there.

3. Between the Two Boundaries

Computation 3.1: **for 400 values of R from just beyond 15.581719 to 200, moving B through 120 values from just outside R out to 10^8, the bi-elliptic transfer was cheaper in every case.At $R = 13$, between the boundaries, the bi-elliptic transfer was cheaper only for B beyond 48.904843, and more expensive for nearer B.**

Observation 3.2

between 11.94 and 15.58, how far out one goes decides which is cheaper.As B approaches R, the bi-elliptic transfer turns into the Hohmann transfer (the third burn becomes 0)——the gain of the long way appears only when one goes far enough.

4. Breakdown and Time

Computation 4.1: **for $R = 20$ and $B = 100$, the Δv of each burn was set out.**

transfer	first	second	third	total
Hohmann	0.380131	0.154600	—	0.534731
bi-elliptic ($B = 100$)	0.407195	0.043663	0.065068	0.515927
bi-elliptic ($B \to \infty$)	—	—	—	0.506835

The bi-elliptic transfer is 0.018805 cheaper. With the period of the starting orbit as 1, the Hohmann transfer takes 17.011944 and the bi-elliptic 411.813965, 24.207343 times as long.

Observation 4.2

the bi-elliptic transfer burns more at the start and a little far out.Its first burn, 0.407195, is larger than the Hohmann 0.380131, while the second, at apoapsis 100, is only 0.043663——it burns most on the starting circle, where the speed is greatest, and changes the shape of the orbit with little at the far apoapsis, where the speed is small.

5. Another Route: Numerical Integration of the Orbits

Computation 5.1: **the orbits were integrated numerically, with burns of the formula's Δv.The Hohmann transfer reached apoapsis 20.000000, and after the second burn the radius stayed 20.000000 through one revolution. The bi-elliptic transfer reached apoapsis 100.000000, periapsis 20.000000 after the second burn, and a radius of 20.000000 through one revolution after the third.Both arrived within 10^{-5}.**

6. Real Sizes

Computation 6.1: **with the Earth's gravitational parameter, the start was a circular orbit at 300 km altitude (radius 6678.137 km).**

target	R	Hohmann	bi-elliptic ($B \to \infty$)
geostationary **42164** km	**6.313737**	**3.892554** km/s	**4.473683** km/s
Moon's distance **384400** km	**57.560963**	**3.936544** km/s	**3.621910** km/s

Observation 6.2

to geostationary orbit (R below 11.94) the Hohmann transfer is cheaper; to the Moon's distance (R beyond 15.58) the bi-elliptic transfer going far out is cheaper.The Hohmann 3.936544 km/s to enter a circular orbit at the Moon's distance is more than the 3.892554 km/s to geostationary orbit——a far more distant target adds only a little Δv, and there the long way wins.

7. The Separator

Observation 7.1

the separator is the ratio of radii R.Below 11.94 the Hohmann transfer wins, beyond 15.58 the bi-elliptic transfer wins for every B, and in between it depends on how far out B is——even between the same two circular orbits, “going straight is cheaper” belongs to a range of R.

Observation 7.2

beside Paper 506.Paper 506 showed that the same fuel yields more energy the faster the craft is moving when it burns——here the bi-elliptic transfer burns more than the Hohmann at the start, where the speed is greatest, and a little at the far apoapsis, where the speed is small. Where one burns shapes the total Δv.

Observation 7.3

what is not said.Real launch planning, time constraints and the gravity of other bodies are not treated, and the gain of the bi-elliptic transfer is not pinned to a single reason——what is shown is the values of the boundaries, the breakdown, and the check by numerical integration.

8. Roll Call and Closing

1. **equal at $R = 11.938765$** (Computation 2.1): **with B ever farther.**
2. **Hohmann largest at $R = 15.581719$** (Computation 2.1): **0.536258.**
3. **dearer than escape from $R = 3.304167$** (Computation 2.1): **exceeding $\sqrt{2} - 1$.**
4. **the Hohmann Δv does not keep rising** (Observation 2.2): **the core.**
5. **beyond 15.58, bi-elliptic for every B** (Computation 3.1): **400 by 120 cases.**
6. **at $R = 13$, only for $B > 48.904843$** (Computation 3.1): **between the boundaries.**
7. **0.018805 cheaper at $R = 20$** (Computation 4.1): **but 24.207343 times as long.**
8. **more at the start, a little far out** (Observation 4.2): **the breakdown.**
9. **numerical integration arrives** (Computation 5.1): **within 10^{-5}.**
10. **where one burns matters** (Observation 7.2): **Paper 506.**

Closing: **Moving from a circular orbit of radius 1 to one of radius R, once R exceeds 11.938765 the bi-elliptic transfer, going farther out first and coming back, needs a smaller total change of velocity than the direct Hohmann transfer, and beyond 15.581719 it is cheaper however far out it goes.The Hohmann Δv does not keep rising with distance: beyond $R = 3.304167$ it costs more than escaping to infinity, and it peaks at 15.581719——so a path going out almost far enough to escape and coming back can be cheaper.That the formula's Δv really arrives was checked by numerical integration of the orbits.The separator is the ratio of radii R.Placed among the earlier papers——the "where one burns" of Paper 506 shows up here in the breakdown of burns.To be honest——time constraints and real launch planning are not entered, and the gain of the bi-elliptic transfer is not pinned to a single reason.**

References

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