The Number of Ways to Write an Even Number as a Sum of Two Primes Differs About Twofold by Whether It Is Divisible by 3

— Every even number from 4 to 1000000 could be written as a sum of two primes — divisibility by 5 gives 1.333126 times and by 7 gives 1.199996 times, each close to $(p-1)/(p-2)$ — [Paper 640]

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— *From thinking to syncing.* —

Abstract

The number of ways $r(n)$ to write an even number n as a sum of two primes $p+q$ ($p \le q$) is counted for everything up to one million.Whether every even number can be so written (Goldbach's conjecture) is not proved. What makes the number of ways rise and fall is examined.No new theorem or law is claimed. First, every even number up to one million could be written——no even number from 4 to 1000000 had 0 ways, and only 4, 6, 8 and 12 had exactly 1. The number of ways is 6 for 100, 127 for 10000 and 5402 for 1000000. The values counted by convolution agreed with direct counting for 310 even numbers (Section 2). **Second, and this is the core.The number of ways rises by fixed factors set by the small primes dividing n——among even numbers from 900000 to 1000000, the average for those divisible by 3 was 1.999939 times that for the rest. Among those not divisible by 3, divisibility by 5 gave 1.333126 times, and among those divisible by neither 3 nor 5, divisibility by 7 gave 1.199996 times, each close to $(p-1)/(p-2)$ (2, 4/3, 6/5)** (Section 3). **Third, the factors come from counting remainders alone——when n is not divisible by 3, for half of the primes q the number $n-q$ is a multiple of 3 and cannot be prime. When n is divisible by 3, no such q exists. So an n divisible by 3 has about twice as many pairs** (Section 3). **Fourth, the conjectured formula fits well——compared with the formula conjectured by Hardy and Littlewood, which includes these factors, for 500 even numbers from 900000 to 1000000 the ratio of the actual count to the formula averaged 0.997559 (from 0.976919 to 1.013639). For 1000000 the actual 5402 stands against 5417.08 from the formula** (Section 4). **Keywords**: Goldbach conjecture, sum of two primes, number of representations, Hardy-Littlewood conjecture, singular series.

Scope of this paper (scope note): **No new theorem or law is claimed——Goldbach's conjecture (a letter to Euler in 1742), the conjectured formula for the number of ways with its factors (Hardy and Littlewood, 1923), and the computational confirmation of the conjecture to far larger numbers (Oliveira e Silva and others, 2014) are known.The conjecture is not proved——it is only counted up to one million, and no amount of counting is a proof.The conjectured formula is only compared, as a conjecture.Relation to earlier papers**: **Paper 621 confirmed that every number up to 10^6 reaches 1 under the Collatz map, which was not a proof——here too Goldbach is only confirmed up to 10^6.Paper 513 showed that counting lattice points turns out to prove quadratic reciprocity——here counting proves nothing, but instead shows that the number of ways carries factors set by the remainders on division by small primes.**What is added is **finding the number of ways by two routes, convolution and direct counting**, **measuring the factors for 3, 5 and 7 in groups with the other primes' effects removed**, and **placing the separator on the small primes dividing n**.

1. Notation and the Question

Setting: **for an even number n, $r(n)$ is the number of pairs $n = p + q$, $p \le q$, with p and q both prime.The primes up to one million were found by a sieve, and convolving the sequence of prime marks with itself (by the fast Fourier transform) counted $r(n)$ for every n at once.**

Question: **what makes the number of ways rise and fall?**

2. Up to One Million

Computation 2.1: **there are 78498 primes up to one million. The $r(n)$ counted by convolution agreed with direct counting for 310 even numbers, 300 chosen at random and 10 representative ones.**

n	4	100	1000	10000	100000	1000000
$r(n)$	1	6	28	127	810	5402

No even number from 4 to 1000000 had $r(n) = 0$. $r(n) = 1$ only for 4, 6, 8 and 12.The smallest prime p with $n - p$ also prime is largest at $n = 503222$, with $p = 523$.

Observation 2.2

up to one million, every even number could be written as a sum of two primes.And, as the table shows, the number of ways grows with n——even so, this is not a proof of the conjecture. It says nothing beyond the range counted.

3. Divisible by 3, 5 or 7

Computation 3.1: among even numbers from 900000 to 1000000, the average number of ways was compared. The factor for 5 was compared among those not divisible by 3, and that for 7 among those divisible by neither 3 nor 5, removing the effect of the earlier primes.

dividing prime	ratio of averages	$(p-1)/(p-2)$	difference
3	**1.999939**	2	−0.000061
5	1.333126	1.333333	−0.000207
7	1.199996	1.200000	−0.000004

Observation 3.2

this is the core: the factors come from counting remainders alone.A prime q larger than 3 leaves remainder 1 or 2 on division by 3. When n is not divisible by 3, for half of the q the number $n-q$ is divisible by 3 and cannot be prime. When n is divisible by 3, $n-q$ is never divisible by 3——so an n divisible by 3 has about twice as many pairs. For 5, whether one of the four remainders of q is lost gives $4/3$; for 7, one of six gives $6/5$.

4. Against the Conjectured Formula

Computation 4.1: the formula conjectured by Hardy and Littlewood,

$$r(n) \approx C_2 \prod_{p \mid n,\, p>2} \frac{p-1}{p-2} \int_2^{n-2} \frac{dt}{\ln t \, \ln(n-t)}$$

(with C_2 the twin prime constant), was compared for 500 even numbers chosen evenly from 900000 to 1000000. The ratio of the actual count to the formula averaged 0.997559, with minimum 0.976919 and maximum 1.013639. For

1000000 the actual 5402 stands against 5417.08 from the formula.

> **Observation 4.2**
>
> **by multiplying in the factors of the small primes, the conjectured formula captures the spread of the number of ways well.Plotting the number of ways against n, the points split into bands by the small primes dividing n (Goldbach's comet)——the ratios between the averages of groups split by dividing primes were the factors close to $(p-1)/(p-2)$ of Computation 3.1.**

5. The Separator

> **Observation 5.1**
>
> **the separator is the small primes dividing n.Among even numbers of the same size, divisibility by 3 gives about twice the ways, by 5 a further $4/3$, and by 7 a further $6/5$——what sets the number of ways is not only the size of n but the line-up of its divisors.**

> **Observation 5.2**
>
> **beside Paper 621 and Paper 513.Paper 621 confirmed the Collatz map up to 10^6, which was not a proof. In Paper 513, counting lattice points proved quadratic reciprocity——here counting does not prove the conjecture. Instead, counting reveals that the number of ways carries factors set by remainders on division.**

> **Observation 5.3**
>
> **what is not said.Goldbach's conjecture is not proved, and the conjectured formula is only compared, not derived——what is shown is the number of ways up to one million and the factors for small primes.**

6. Roll Call and Closing

1. **every even number up to one million can be written** (Computation 2.1): **none with 0 ways.**
2. **only $4, 6, 8, 12$ have one way** (Computation 2.1): **two or more beyond.**
3. $r(1000000) = 5402$ (Computation 2.1): **two ways of counting agree.**

4. **the largest least prime is 523** (Computation 2.1): **at $n = 503222$.**
5. **counting is not proof** (Observation 2.2): **nothing beyond the range.**
6. **1.999939 times when divisible by 3** (Computation 3.1): **close to 2.**
7. **1.333126 for 5, 1.199996 for 7** (Computation 3.1): **$4/3$ and $6/5$.**
8. **the factors come from remainders** (Observation 3.2): **the core.**
9. **average ratio 0.997559 to the formula** (Computation 4.1): **over 500 numbers.**
10. **on the side where counting does not prove** (Observation 5.2): **Paper 621 and Paper 513.**

Closing: **Every even number from 4 to 1000000 could be written as a sum of two primes, and only 4, 6, 8 and 12 in just one way.The number of ways rises by fixed factors set by the small primes dividing n: from 900000 to 1000000, 1.999939 times for divisibility by 3, 1.333126 for 5 and 1.199996 for 7——counting remainders alone gives the factor $(p-1)/(p-2)$.The conjectured formula including these factors fits the actual counts with an average ratio of 0.997559.The separator is the small primes dividing n.Placed among the earlier papers——as with Paper 621, it is only confirmed up to 10^6; unlike Paper 513, counting proves nothing here.To be honest——neither a proof of Goldbach's conjecture nor a derivation of the conjectured formula is entered.**

References

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