

The Roads to Fermat's Last Theorem Reach Different Exponents, and Descent and Kummer Together Miss 92 Exponents up to 1000

— together the two still miss **92** exponents up to **1000** (the smallest is **37**) — only Wiles's road reached them all

— [Paper 642]

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— *From thinking to syncing.* —

Abstract

For $n \geq 3$, $x^n + y^n = z^n$ has no solution in coprime positive integers (Fermat's Last Theorem, proved by Wiles in 1995). This paper proves nothing; it sets out the roads that reach this “no solution” and counts which exponents each reaches. No new theorem or law is claimed. First, the road of infinite descent reaches $n = 4$ — $x^4 + y^4 = z^2$ has no solution up to 3000, while the same search found 534 coprime solutions of $x^2 + y^2 = z^2$. If the exponent is divisible by 4, this road suffices (Section 3). Second, Kummer's road reaches the regular primes—if a prime p is regular (p divides none of the numerators of the Bernoulli numbers $B_2, \dots, B_{p-3}$), there is no solution with exponent p . Of the 167 primes from 3 to below 1000, 103 are regular (a share of 0.616766), and the only irregular ones below 100 are 37, 59 and 67 (Section 4). Third, and this is the core. Even together, the two roads leave exponents unreachable—exponents not divisible by 4 and with no regular odd prime factor number **92** up to 1000, the smallest being 37. Each tool reaches a different range of exponents, and only Wiles's road reached them all (Section 5). Fourth, the road from the ABC conjecture reaches large exponents (conditionally)—a counterexample would have quality $\log z^n / \log \text{rad}(xyz)$ above $n/3$. For $n \geq 5$ that exceeds $5/3$, above the known highest, 1.629912, checked in Paper 641. None of the 1268 triples with $c < 10^6$ has $q > 5/3$. Triples exceeding the lower bounds 1 and $4/3$ for $n = 3$ and 4, however, do exist (Section 6). Fifth, in the world of polynomials it takes one line, and near misses are not solutions—an exhaustive search over polynomials with small coefficients found 20 pairs for $n = 2$ and none for $n = 3$ or 4 (a consequence of Mason--Stothers). Among integers there are 18 triples up to 2000 that miss by one, such as $9^3 + 10^3 = 12^3 + 1$, but none that match exactly (Sections 2 and 7). **Keywords:** Fermat's Last Theorem, infinite descent, regular primes, Bernoulli numbers, Kummer.

Scope of this paper (scope note): No new theorem or law is claimed—the Last Theorem and its proof (Wiles, and Taylor and Wiles, 1995), infinite descent for $n = 4$ (Fermat), Kummer’s theorem on regular primes (around 1850), the list of irregular primes, the conjecture that the share of regular primes is near $e^{-1/2}$ (Siegel), and that the ABC conjecture excludes large exponents (the exposition of Granville and Tucker and others) are all known. The Last Theorem is not proved—Wiles’s road (the Frey curve and modularity) cannot be followed in numbers, so only its address is written. It is not called “distinct roots”—what divides here is not the source of a phenomenon but the tool of proof, so it is called “distinct roads”. Theorems with the same name are kept apart—Fermat’s test (the little theorem) in Paper 603 and Kummer’s theorem on carries in Paper 608 are different theorems from the Fermat and Kummer of this paper. Relation to earlier papers: Paper 641 counted that the ABC conjecture is a theorem for polynomials and a conjecture for integers, and placed Fermat for polynomials as a one-line consequence—this paper places that polynomial road and the road from ABC within a roll call of roads to the integer Last Theorem. Paper 31 gave the π of Ramanujan’s series the address of a period of modular forms—Wiles’s road also passes through modular forms. What is added is deciding regularity up to 1000 by two routes, counting the exponents reached by neither descent nor Kummer, matching the lower bound $n/3$ on a counterexample’s quality with known triples, and placing the separator on the range of exponents each tool reaches.

1. Notation and the Question

The Last Theorem: for $n \geq 3$, $x^n + y^n = z^n$ has no solution in coprime positive integers. For a divisor d of n , a solution of $x^n + y^n = z^n$ gives a solution of $(x^{n/d})^d + (y^{n/d})^d = (z^{n/d})^d$. Every $n \geq 3$ is divisible by 4 or by an odd prime, so it suffices to show there is no solution for $n = 4$ and for odd primes p .

Question: which tools reach this “no solution”, and up to which exponents?

2. The World of Polynomials

Computation 2.1: from the 124 polynomials of degree at most 2 with integer coefficients from -2 to 2 , coprime non-constant pairs $x(t), y(t)$ were chosen, and those for which $x^n + y^n$ is the n -th power of a polynomial were sought (narrowed by the values at $t = 2$ to 8 and confirmed by factorisation). For $n = 2$, 20 pairs were found (including $(2t)^2 + (t^2 - 1)^2 = (t^2 + 1)^2$); for $n = 3$ and 4 , none.

Observation 2.2

in the world of polynomials, every $n \geq 3$ is reached in one line. Applying the Mason--Stothers theorem of Paper 641, with D the largest degree, gives $nD \leq 3D - 1$, so $n < 3$ —but this is the Last Theorem for polynomials, not a road to the Last Theorem for integers. Integers lack the derivative that its proof uses.

3. The Road of Infinite Descent (n = 4)

Computation 3.1: searching $1 \leq x \leq y \leq 3000$ for an integer z with $x^4 + y^4 = z^2$ found none. The same search found 534 coprime solutions of $x^2 + y^2 = z^2$ (a positive control that the search works).

Observation 3.2

Fermat’s own road denies a solution by showing that any solution would yield a smaller one (infinite descent). Since $x^4 + y^4 = z^4$ is a special case of $x^4 + y^4 = (z^2)^2$, this settles $n = 4$ —and every exponent divisible by 4 reduces to $n = 4$.

4. Kummer’s Road (Regular Primes)

Computation 4.1: an odd prime p is regular when p divides none of the numerators of the Bernoulli numbers $B_2, B_4, \dots, B_{p-3}$. Regularity was decided by two routes, one using exact Bernoulli numbers and one following only remainders on division by p through a recurrence, and the results agreed for every prime $p < 200$. Regularity was then decided up to $p < 1000$.

primes from 3 to below 1000	167
regular	103 (share 0.616766)
irregular	64
irregular primes below 100	37, 59, 67

Primes of irregularity index 2 or more include 157 ($k = 62, 110$) and 691 ($k = 12, 200$), 14 in all; 491, 617 and 647 have index 3. The share 0.616766 differs by +0.010236 from Siegel’s conjectured $e^{-1/2} = 0.606531$.

Observation 4.2

Kummer’s road reaches only the regular primes. And while it is known that there are infinitely many irregular primes, whether there are infinitely many regular primes is still unknown—the very breadth of this road’s reach is an open question.

5. Exponents Neither Road Reaches

Computation 5.1: among the exponents $n = 3$ to 1000, those not divisible by 4 and with no regular odd prime factor (reducing neither to infinite descent nor to Kummer’s road) number 92. In increasing order they run 37, 59, 67, 74, 101, 103, 118, 131, 134, 149. In each, every odd prime factor is irregular, and none is divisible by 4.

Observation 5.2

this is the core: each tool reaches a different range of exponents, and even together two of them leave 92 gaps up to 1000. Infinite descent reaches the multiples of 4, Kummer’s road the exponents with a regular odd prime factor. An irregular prime such as 37 itself is reached by neither—the same “no solution” was reached by different roads for different exponents, and only Wiles’s road (the Frey curve and modularity) reached them all.

6. The Road from the ABC Conjecture (Conditional)

Computation 6.1: if coprime $x < y < z$ satisfy $x^n + y^n = z^n$, then $\text{rad}(x^n y^n z^n) = \text{rad}(xyz) \leq xyz < z^3$, so the quality of this triple satisfies

$$q = \frac{\log z^n}{\log \text{rad}(xyz)} > \frac{n}{3}.$$

$n/3$ is 1.000000 for $n = 3$, 1.333333 for 4, 1.666667 for 5, 2.000000 for 6 and 2.333333 for 7.

known highest quality ($2 + 3^{10} \times 109 = 23^5$, Paper 641)		1.629912 (difference from 5/3: -0.036755)
triples with $q > 1$ and $c < 10^6$		1268 (highest 1.567887)
of which $q > 4/3$		20
of which $q > 5/3$		0

Observation 6.2

for $n \geq 5$, a counterexample would be a triple of higher quality than any found so far. If the ABC conjecture is true, only finitely many triples have quality above $5/3$, so counterexamples with large exponents are limited. On the other hand, triples exceeding the lower bounds 1 and $4/3$ for $n = 3$ and 4 exist even with $c < 10^6$, and this road alone cannot exclude them—the road from ABC reaches large exponents, and infinite descent and Kummer’s road take care of the small ones.

7. Near but Not There

Computation 7.1: with $2 \leq x \leq y \leq 2000$, 18 triples have $x^3 + y^3$ differing from a cube by one (9 on the side $z^3 + 1$ and 9 on the side $z^3 - 1$). In increasing order they begin $6^3 + 8^3 = 9^3 - 1$, $9^3 + 10^3 = 12^3 + 1$ (1729) and $64^3 + 94^3 = 103^3 + 1$. No triple with $x^3 + y^3 = z^3$ was found.

Observation 7.2

near misses are no clue to a solution. There are 18 triples up to 2000 that miss by one, but none with a miss of 0—“no solution” is not a matter of closeness but of a structure that cannot be exactly equal.

8. The Separator

Observation 8.1

the separator is the range of exponents each tool reaches. Infinite descent reaches the multiples of 4, Kummer’s road the regular primes, the road from ABC (conditionally) the large exponents, and the polynomial road the world of polynomials. Each is partial, and 92 gaps remain up to 1000—the same “no solution” is supported by different tools over their own ranges, and only Wiles’s road supported all of it.

Observation 8.2

beside Paper 641, Paper 31, Paper 603 and Paper 608. Paper 641 is where the polynomial road and the road from ABC come from, and Paper 31 wrote an address in modular forms. Fermat's test in Paper 603 and Kummer's theorem in Paper 608 are different theorems by the same people—the same name, pointing to a different theorem, is a different tool.

Observation 8.3

what is not said. The Last Theorem is not proved, only the address of Wiles's road is written, and it is not called "distinct roots"—what is shown is up to which exponents each of four roads reaches, and how many exponents they miss.

9. Roll Call and Closing

1. no polynomial solutions for $n = 3$ and 4 (Computation 2.1): **20 pairs for $n = 2$.**
2. the polynomial road is not an integer road (Observation 2.2): **no derivative.**
3. $x^4 + y^4 = z^2$: none up to 3000 (Computation 3.1): **positive control 534.**
4. regular primes: **103 of 167** (Computation 4.1): **0.616766.**
5. irregular primes below 100: **37, 59, 67** (Computation 4.1): **two routes agree.**
6. infinitely many regular primes is open (Observation 4.2): **the road's breadth is a question.**
7. **92** exponents reached by neither road (Computation 5.1): **the smallest is 37.**
8. each tool has its own range (Observation 5.2): **the core.**
9. a counterexample's quality exceeds $n/3$ (Computation 6.1): **beyond the known highest for $n \geq 5$.**
10. **18** near misses, **0** exact (Computation 7.1): **closeness is no clue.**

Closing: To Fermat's Last Theorem, that $x^n + y^n = z^n$ has no solution for $n \geq 3$, infinite descent reaches from $n = 4$ and Kummer's road from the regular primes, and 0.616766 of the odd primes below 1000 were regular. Even together the two do not reach 92 of the exponents up to 1000 (the smallest being 37)—since a counterexample's quality would exceed $n/3$, the road from the ABC conjecture reaches large exponents conditionally, but not the small ones. In the world of polynomials it takes one line, and integer triples that miss by one are not solutions. The separator is the range of exponents each tool reaches. Placed among the earlier papers—the polynomial road and

the ABC road of Paper 641 enter the roll call of roads to the integer Last Theorem. To be honest——Wiles’s road is given only an address, and the proof of the Last Theorem is not entered.

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