

Velocity-Space Curvature and Gravitational Redshift

— Why "Local c " and "Coordinate c " Are Distinct Roots —

Resolving, Negatively, an Open Problem of Paper 63

Author: Yuuki YAMAGISHI (山岸 勇輝)

Affiliation: Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

Positioning of this paper

This paper offers a **structural interpretation** of established relations. It does not modify general relativity, does not propose a variable speed of light, and claims no new physical law. Its result is a clarification: the relation $E = 4\pi/|K|$ of Paper 60 is a *local* invariant and does not—and should not—reproduce gravitational redshift, because "local c " (the radius of velocity space) and "coordinate c " (a gauge quantity of the base metric) are **distinct**. This resolves, negatively, an open problem posed in Paper 63. No claim contradicting established theory is made.

ABSTRACT

Paper 60 showed that, in the $m = 4\pi$ representation, the rest energy is $E = 4\pi/|K|$ with $K = -1/c^2$ the sectional curvature of the hyperbolic velocity space. Paper 63 asked, as an open problem, whether substituting the position-dependent coordinate light speed $K_{\text{eff}}(r) = -1/c_{\text{coord}}(r)^2$ into $E_{\text{eff}} = 4\pi/|K_{\text{eff}}|$ reproduces the gravitational redshift. We answer **negatively**, and the manner of failure is sharp. (i) In Schwarzschild spacetime the coordinate light speed is *direction-dependent*— $c(1 - r_s/r)$ radially, $c\sqrt{1 - r_s/r}$ transversely—so it cannot be the radius of any velocity ball. (ii) The naive substitution gives an energy factor $(1 - r_s/r)$ (transverse) or $(1 - r_s/r)^2$ (radial), whereas the true redshift factor is $\sqrt{1 - r_s/r}$; it over-applies the metric factor by one power of $\sqrt{g_{00}}$. (iii) Correctly, by the equivalence principle (fiber constancy, Paper 63) the local velocity space is a hyperbolic **3**-space of curvature $-1/c^2$ at *every* event, so $E = 4\pi/|K| = 4\pi c^2$ is a local scalar invariant; redshift lives in the base metric, $E_{\infty} = E_{\text{local}}\sqrt{g_{00}}$, i.e. in the connection *between* fibers. We conclude that "local c " (fiber radius) and "coordinate c " (base gauge) are **distinct roots**, the c -analogue of the factor-two result of Paper 58. The variable-speed-of-light temptation is precisely their conflation.

1. Introduction

Paper 60 established that, in velocity space—the hyperbolic **3**-ball $\{|v| < c\}$ of curvature $K = -1/c^2$ (Varičák 1910)—the rest energy is $E = 4\pi c^2 = 4\pi/|K|$. This is the one place where " c is a radius" holds literally rather than as a notational patch. Paper 63 paired this with the Einstein equation as a curvature duality and asked, as an open problem, whether a curved spacetime—where the coordinate speed of light varies—would make $E_{\text{eff}} = 4\pi/|K_{\text{eff}}|$ reproduce the gravitational redshift.

This paper answers that question. The answer is *no*, and the reason is instructive: **the quantity that varies (the coordinate light speed) is not the quantity that plays the role of a radius (the local light speed)**. Making this precise resolves the open problem and sharpens the meaning of " c as a radius."

What this paper does not do. It does not alter general relativity or advocate a varying speed of light. It clarifies which c is which.

2. Recap: Two Established Ingredients

2.1 $E = 4\pi/|K|$ (Paper 60)

In special relativity the admissible velocities $\{|v| < c\}$ form a hyperbolic **3**-space $\mathbb{H}^3$ of curvature $K = -1/c^2$; under $m = 4\pi$, $E = 4\pi c^2 = 4\pi/|K|$. The curvature $|K|$ is measured as Thomas precession.

2.2 Fiber constancy (Paper 63)

By the equivalence principle, the tangent space $T_p M$ at every event p is Minkowskian. Hence the local velocity space is the *same* hyperbolic **3**-space of curvature $-1/c^2$ at every event (the vertical/fiber expression of the equivalence principle): the base curvature $R_{\mu\nu\rho\sigma}$ varies, but the fiber curvature $K = -1/c^2$ is uniform.

3. The Tempting Substitution and Its Failure

3.1 The coordinate light speed is direction-dependent

In Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

null geodesics give two different "coordinate speeds of light":

$$c_{\text{coord}}^{\text{radial}} = c \left(1 - \frac{r_s}{r}\right), \quad c_{\text{coord}}^{\text{transverse}} = c \sqrt{1 - \frac{r_s}{r}}$$

the first from $d\mathbf{r}/dt$ on a radial null ray, the second from $r d\phi/dt$ on a transverse one. **This alone is fatal: a radius is a single number, but c_{coord} depends on direction. No sphere has a direction-dependent radius.**

3.2 The energy factor overshoots the redshift

Set $f := 1 - r_s/r$. The gravitational redshift of energy (a locally emitted quantum received at infinity) is

$E_\infty/E_{\text{local}} = \sqrt{g_{00}} = \sqrt{f}$. The naive substitution $E_{\text{eff}} = 4\pi c_{\text{coord}}^2$, however, gives:

$$\frac{E_{\text{eff}}^{\text{transverse}}}{4\pi c^2} = f, \quad \frac{E_{\text{eff}}^{\text{radial}}}{4\pi c^2} = f^2$$

Proposition 3.1 (The substitution over-applies the metric factor)

Relative to the true redshift factor $\sqrt{f}$, the naive substitution overshoots by an extra power of $\sqrt{f}$ in the transverse case (factor $f = \sqrt{f} \cdot \sqrt{f}$) and by $f^{3/2}$ in the radial case. In neither case is the redshift reproduced.

The origin is simple: $E = 4\pi c^2$ carries c *squared*, so replacing c by $c_{\text{coord}} = c\sqrt{f}$ multiplies by f —the redshift factor applied *twice*—whereas the physical redshift applies $\sqrt{f}$ *once*, through the single factor $\sqrt{g_{00}}$.

4. The Correct Picture

Theorem 4.1 (Local invariance of the velocity-space curvature)

In any spacetime, at each event p the set of future-directed unit timelike vectors in $T_p M$, with the local Minkowski metric guaranteed by the equivalence principle, is a hyperbolic 3-space $\mathbb{H}^3$ of constant sectional curvature $K = -1/c^2$. Consequently $E = 4\pi/|K| = 4\pi c^2$ is a scalar invariant, identical at every event, with the *local* c .

Proof. The equivalence principle states that $T_p\mathcal{M}$ is isometric to Minkowski space at every p . The space of unit future timelike vectors in Minkowski space is the unit hyperboloid, i.e. $\mathbb{H}^3$ with curvature $-1/c^2$ (Varičák 1910). This holds at every p with the same local c ; hence $|K| = 1/c^2$ and $E = 4\pi/|K| = 4\pi c^2$ everywhere. ■

Corollary 4.2 (Redshift is a base effect, not a fiber effect)

The energy at infinity of a locally measured energy is $E_\infty = E_{\text{local}}\sqrt{g_{00}} = 4\pi c^2 \sqrt{1 - r_s/r}$. The factor $\sqrt{g_{00}}$ arises from the relation between local proper time and coordinate time—how fibers are glued by the connection—not from any change in the fiber curvature K , which is constant by Theorem 4.1. Therefore $E = 4\pi/|K|$ neither encodes nor should encode gravitational redshift.

5. Distinct Roots for c

The situation is the exact c -analogue of Paper 58 (two factors of 2 , equal at $D = 4$, with distinct origins). Here two objects both called " c " must be separated:

	Local c	Coordinate c
Habitat	fiber $T_p\mathcal{M}$ (velocity space)	base metric (gauge)
Nature	invariant; single number	direction-dependent; gauge
Role	radius of $\mathbb{H}^3$; gives $E = 4\pi c^2$	bookkeeping in chosen coordinates
Encodes redshift?	no (constant)	no (not a measurement)
Redshift lives in	—	$\sqrt{g_{00}}$ (connection between fibers)

The two coincide only in the flat/local limit, exactly as $f_{\text{GR}}(D)$ and $f_{\text{ladder}}(D)$ coincide only at $D = 4$ in Paper 58. The variable-speed-of-light programmes (Moffat 1993; Albrecht & Magueijo 1999; critique in Ellis & Uzan 2005) are tempting precisely because they conflate the two. This analysis shows that " c as the radius of velocity space" (Paper 60) is a statement about the local c only, and is therefore untouched by, and orthogonal to, the coordinate- c variation of a curved spacetime.

6. Scope of the Claims

What is established

1. The naive substitution $E_{\text{eff}} = 4\pi c_{\text{coord}}^2$ does not reproduce redshift; it overshoots by a power of $\sqrt{g_{00}}$ and is ill-defined because c_{coord} is direction-dependent (Proposition 3.1).
2. $E = 4\pi/|K| = 4\pi c^2$ is a local scalar invariant, identical at every event (Theorem 4.1).
3. Redshift is a base-metric (connection) effect, $E_{\infty} = E_{\text{local}}\sqrt{g_{00}}$, orthogonal to the fiber curvature (Corollary 4.2).

What is not established / Open problem

- Any modification of general relativity or of the redshift formula (both are used, not changed).
- Any variable speed of light (the paper argues *against* conflating coordinate and local c).
- A new derivation of redshift (it is taken as given; the contribution is the fiber/base separation and the "distinct roots for c " reading).
- *Open*: is there a natural pairing on TM (or T^*M) in which the base factor $\sqrt{g_{00}}$ and the fiber invariant $4\pi c^2$ combine covariantly into the true E_{∞} ? This is the covariant version of the hinge of Paper 63.

7. Conclusion

The velocity-space curvature $K = -1/c^2$ does not redshift. By the equivalence principle it is a local invariant, so $E = 4\pi/|K| = 4\pi c^2$ is the same at every event; gravitational redshift is carried by the base metric factor $\sqrt{g_{00}}$, the connection between fibers. The tempting substitution of the coordinate light speed fails—overshooting the redshift and, worse, being direction-dependent—because it confuses the coordinate c (a gauge quantity of the base) with the local c (the radius of the fiber). "Local c " and "coordinate c " are distinct roots, the c -analogue of the factor-two result of Paper 58. This both resolves, negatively, the open problem of Paper 63 and sharpens " c as a radius" into a statement about the local c alone.

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Yuuki YAMAGISHI (山岸 勇輝)

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