

The Euler-Characteristic Origin of the 4π Patch

Reading $m = 4\pi$ as $2\pi\chi(S^2)$ via Poincaré–Hopf

— A Topological Anchor for the Yamagishi Solid-Angle Constant — [Paper 67]

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— *From thinking to syncing.* —

Abstract

The constant 4π —the total solid angle of the 2 -sphere and the limit mass $m = 4\pi$ of Paper 01—anchors the entire Yamagishi system. This paper observes that 4π is not merely a geometric quantity but a **topological** one: by the Gauss–Bonnet theorem, $\int_{S^2} K dA = 2\pi\chi(S^2)$, and by the Poincaré–Hopf theorem the Euler characteristic $\chi(S^2) = 2$ is fixed by topology alone (the total index of any vector field on S^2 is 2 ; “one cannot comb the hairy ball”). Hence

$$4\pi = \int_{S^2} K dA = 2\pi\chi(S^2) = 2\pi \times 2, \quad \text{so} \quad m = 4\pi = 2\pi\chi(S^2).$$

The “ 2 ” in $4\pi = 2\pi \cdot 2$ is precisely the Euler characteristic of the sphere. We record three consequences: (i) the cornerstone constant of the system has a topological origin, so its value is forced, not chosen; (ii) the topological anchor exists only at *even* sphere dimensions, since $\chi(S^n) = 1 + (-1)^n$ vanishes for odd n —echoing the even/odd split of Paper 57; (iii) in higher even dimensions the same anchor is the Chern–Gauss–Bonnet theorem, connecting to the $(4\pi)^n n!$ normalisation of Paper 48. We are explicit that this is a re-reading of a known constant, not a derivation of new physics, and that the metric-independent invariant is the curvature integral $\int K dA$, which the round unit sphere realises as its area 4π .

Keywords: solid angle, Euler characteristic, Poincaré–Hopf theorem, Gauss–Bonnet, hairy ball theorem, 4π , topological invariant, structural interpretation.

1. Introduction

Paper 01 established the limit mass $m = 4\pi$, the pure total solid angle of three-dimensional space, as the anchor of the Yamagishi system. A natural question is whether the value 4π is arbitrary—a mere geometric happenstance—or whether it is forced. This paper answers: it is **forced, and the force is topological**.

The route is deliberate. In earlier work (research notes on Paper 15) a tempting bridge from the atomic “ $16\pi^2$ ” to the instanton “ $16\pi^2$ ” of the Hopf fibration was examined and found to *fail*: mass and spin are independent Casimirs of the

Poincaré group, so the two $16\pi^2$ are distinct roots. The present paper therefore reaches Hopf through a different, sound door: the **Euler characteristic**, via the Poincaré–Hopf theorem. Through this door the connection is not a coincidence of numbers but an identity of a well-defined topological invariant.

What this paper does not do. It does not alter physics, does not derive that mass “must” be 4π (the solid-angle representation remains a choice), and predicts nothing new. It shows that the *value* of the solid angle is topological.

2. Three Established Facts

2.1 The total solid angle is 4π

The total solid angle subtended at a point in $\mathbb{R}^3$ is

$$\Omega = \int_{S^2} d\Omega = \text{area}(S^2_{\text{unit}}) = 4\pi, \quad (1)$$

the area of the unit **2**-sphere. This is the $m = 4\pi$ of Paper 01.

2.2 Gauss–Bonnet: curvature integral = $2\pi\chi$

For a closed oriented Riemannian surface M ,

$$\int_M K dA = 2\pi \chi(M), \quad (2)$$

where K is the Gaussian curvature and $\chi(M)$ the Euler characteristic. For the unit sphere $K \equiv 1$, so $\int_{S^2} K dA = \text{area}(S^2) = 4\pi$, giving $2\pi\chi(S^2) = 4\pi$.

2.3 Poincaré–Hopf: $\chi(S^2) = 2$ is topological

The Poincaré–Hopf theorem states that for any smooth vector field v with isolated zeros on a compact manifold M ,

$$\sum_{p: v(p)=0} \text{index}(v; p) = \chi(M). \quad (3)$$

For S^2 the total index is **2**: *every* vector field on the sphere must vanish somewhere (the “hairy ball theorem”—one cannot comb the hair on a sphere without a cowlick), and the indices sum to **2**. Crucially, (3) is independent of any metric: $\chi(S^2) = 2$ is fixed by topology alone.

3. The Synthesis

Proposition 3.1 (Topological value of the solid-angle patch)

The total solid angle of the **2**-sphere satisfies

$$4\pi = \int_{S^2} K dA = 2\pi \chi(S^2) = 2\pi \times 2, \quad (4)$$

where $\chi(S^2) = 2$ is fixed by the Poincaré–Hopf theorem (3) independently of the metric. Consequently the Yamagishi patch is

$$\boxed{m = 4\pi = 2\pi \chi(S^2)}. \quad (5)$$

Proof.

By Poincaré–Hopf (3), $\chi(S^2) = 2$. By Gauss–Bonnet (2), $\int_{S^2} K dA = 2\pi \chi(S^2) = 4\pi$. For the round unit sphere $K \equiv 1$, the curvature integral equals the area (1), which is the total solid angle 4π . Hence $4\pi = 2\pi \chi(S^2)$ and $m = 4\pi = 2\pi \chi(S^2)$. ■

The content is a reading: the “**2**” in $4\pi = 2\pi \cdot 2$ is the *Euler characteristic of the sphere*, and it is a topological number, not a geometric accident. The round sphere ($K \equiv 1$) is the representative for which the metric-independent invariant $\int K dA = 2\pi \chi$ is realised as the area 4π , i.e. as the total solid angle.

4. Even/Odd Dimensions and the Higher Ladder

The Euler characteristic of the n -sphere is

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 2, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases} \quad (6)$$

Thus the topological anchor of Proposition 3.1 exists *only at even sphere dimensions*; at odd dimensions the Euler characteristic vanishes. This echoes the even/odd split observed in Paper 57, where $(4\pi)^{d/2}$ carries integer powers of π at even d and half-integer powers at odd d : the “clean” topological content sits at even dimensions in both readings.

Corollary 4.1 (Higher-dimensional anchor)

In $2n$ dimensions the analogue of (2) is the Chern–Gauss–Bonnet theorem,

$$\int_{M^{2n}} \text{Pf}\left(\frac{\Omega}{2\pi}\right) = \chi(M^{2n}),$$

whose normalisation is the $(4\pi)^n n!$ constant unpacked in Paper 48. The topological reading of 4π given here is the $d = 2$ base case of that hierarchy.

We do not claim more than the arithmetic identity here: as in Paper 48, distinct normalisation constants are not to be casually identified. The statement is only that the base of the ladder, 4π , is $2\pi\chi(S^2)$.

5. Scope of the Claims

What is established:

1. $4\pi = 2\pi\chi(S^2)$ with $\chi(S^2) = 2$ fixed topologically by Poincaré–Hopf; hence $m = 4\pi = 2\pi\chi(S^2)$ (Proposition 3.1).
2. The topological anchor exists only at even sphere dimensions, $\chi(S^n) = 1 + (-1)^n$, echoing Paper 57 (Section 4).
3. The higher-dimensional anchor is Chern–Gauss–Bonnet, whose $(4\pi)^n n!$ normalisation is Paper 48 (Corollary 4.1).

What is not established / not claimed:

- Any new physics or prediction. This is a topological re-reading of a known constant.
- That mass “must” be 4π . The solid-angle representation ($m = 4\pi$) remains a representational choice; only the *value* of the solid angle is shown to be topological.
- Any identification via the mass–instanton “ $16\pi^2$ ” bridge, which remains distinct (Paper 15 note). This paper uses only the Euler-characteristic door.

Remark on the two “ 4π ”. The solid angle $\int d\Omega$ (area of the round unit sphere) and the topological quantity $2\pi\chi(S^2)$ coincide because the round sphere has $K \equiv 1$; the metric-independent invariant is the curvature integral $\int K dA$. It is in this precise sense that the patch value 4π is topological.

6. Conclusion

The constant 4π that anchors the Yamagishi system is $2\pi\chi(S^2)$: the total solid angle equals 2π times the Euler characteristic of the sphere, and $\chi(S^2) = 2$ is fixed—independently of any metric—by the Poincaré–Hopf theorem. The cornerstone value is therefore topological, not arbitrary, and the “2” in $4\pi = 2\pi \cdot 2$ is the Euler characteristic. The anchor is

present only at even dimensions ($\chi(S^n) = 1 + (-1)^n$) and generalises, via Chern–Gauss–Bonnet, to the $(4\pi)^n n!$ ladder of Paper 48. This reaches Hopf through the Euler-characteristic door, keeping clear of the distinct mass–instanton $16\pi^2$. No new physics is claimed; a known constant is given a topological reading.

Revision Record

Second edition (September 2026): no claim, theorem or check has been altered from the first edition. German, French and Italian versions have been added. The references that cite earlier papers of this corpus now give **the actual titles of the cited papers in the current format** (the paper number and DOI after the title); the first edition had shortened the titles. The paper number has been moved onto the title line, and the DOI has been removed from the date line. Paper 15, cited in the text, has been added to the references. **Nothing of substance has changed, and nothing has been withdrawn.**

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