What Decides Between “Two Tones” and “One Wavering Tone” Is Not the Sound but the Listener

— The sum of two sine waves is an identity, and the beat rate is settled inside the signal (the sum-to-product rewriting agreed to **12** digits) — the separator is whether the frequency difference exceeds the critical band — [Paper 672]

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— From thinking to syncing. —

Abstract

Sound two tones together and sometimes two tones are heard, sometimes one tone wavering. Where is the boundary?The quantities on the side of the beat and on the side of the hearing are counted side by side.No new theorem or law is claimed. First, the beat rate is settled on the side of the sound——the sum of two sine waves can be rewritten as an envelope times a mean tone; checked numerically it agrees to 12 digits, and although the envelope’s period is half the difference, the peaks in strength come at 3 per second, the full difference (Section 2). **Second, and this is the core. The quantity that sets the boundary belongs to the listener——the critical bandwidth at 440 Hz is 113.4975 Hz by Zwicker’s formula and 72.1932 Hz by the equivalent rectangular bandwidth, 1.572137 times apart, both being standard** (Section 3). **Third, in musical terms that band is wide——Zwicker’s band at 440 Hz is 397.2880 cents, that is 3.972880 semitones, and even the equivalent rectangular bandwidth is 263.0214 cents** (Section 3). **Fourth, the same beat means different intervals at different pitches——a 1 Hz beat is 15.667383 cents at 110 Hz, 3.930158 at 440 Hz and 1.966194 at 880 Hz, the ends standing 7.968379 times apart** (Section 2). **Fifth, the cents of an earlier paper become beats per second——the 13.686286 cent error of the equal-tempered major third shown in Paper 466 is 4.365262 beats per second above 440 Hz, and the 1.955001 cent error of the fifth is 0.744886 beats per second** (Section 4). **Sixth, the separator is whether the frequency difference exceeds the critical band——the same two tones are heard as two if it does and as one wavering tone if it does not** (Section 4).
Keywords: beats, critical bandwidth, equivalent rectangular bandwidth, cent, equal temperament, frequency resolution.

Scope of this paper (scope note): **No new theorem or law is claimed——the sum-to-product identity behind beats, the critical bandwidth (Zwicker 1961), the equivalent rectangular bandwidth (Glasberg and Moore 1990) and the definition of the cent are all standard.No model of hearing is built——two formulas for the critical bandwidth are taken from the literature, and no judgement is made as to which is right.Consonance is not discussed——why some intervals sound smooth and others rough is not the subject.Temperaments are not evaluated——which temperament is beautiful, and how one should tune, are not the subject (the same line as Paper 466).Individual differences are not treated——loudness, timbre, training and age are not included.Relation to earlier papers**: **Paper 466 showed by the arithmetic of ratios that perfect fifths cannot be stacked into octaves, and stated that it did not discuss hearing: beats, the limits of audibility and consonance were not its subject, only the ratios——this paper takes that seat and turns the cents of Paper 466 into beats per second.Paper 649 treated tide tables and showed that the beat of two tidal constituents is 14.7653 days and that separating them needs a record of that length——there the beat was a period in the signal; here the same beat is seen from the side that must separate it.**What is added is **checking the identity numerically**, **setting two standard definitions of the critical bandwidth side by side and showing them 1.572137 times apart**, **deriving how the same 1 Hz becomes different intervals with pitch**, **turning the cents of Paper 466 into beat rates**, and **placing the separator on the critical band**.

1. Notation and the Question

Setting: **two tones of frequencies f_1 and f_2 sound together. Their sum can be rewritten as**

$$\sin(2\pi f_1 t) + \sin(2\pi f_2 t) = 2\cos\left(2\pi \frac{f_2 - f_1}{2} t\right) \sin\left(2\pi \frac{f_1 + f_2}{2} t\right)$$

Intervals are measured in cents, 1200 to the octave.On the side of hearing, the critical bandwidth is used.

Question: **is the difference between “two tones” and “one wavering tone” a quantity of the sound or of the listener?**

2. The Side of the Sound Is Settled by an Identity

Computation 2.1: **for 440 Hz and 443 Hz the rewriting was checked numerically: at one instant the left side is -1.486883311274 and the right side -1.486883311274, a difference of 4.7×10^{-14}.**

Observation 2.2

the beat rate is fixed before anyone listens.The envelope’s period is half the difference (1.5 Hz), but peaks in strength come twice per envelope cycle, so 3 per second——all of this is a property of the signal, with no listener in it.

Computation 2.3: **taking 440 Hz as the reference, beat rates translate into intervals as follows.**

beat	interval
0.5 Hz	**1.966194** cents
1 Hz	**3.930158** cents
2 Hz	**7.851415** cents
5 Hz	**19.562175** cents

The same 1 Hz beat is 15.667383 cents at 110 Hz, 7.851415 cents at 220 Hz and 1.966194 cents at 880 Hz, the ends standing 7.968379 times apart.

3. The Side of the Listener Moves with the Definition

Computation 3.1: **the critical bandwidth was computed from two standard formulas.**

centre frequency	Zwicker	equivalent rectangular	ratio
220 Hz	**103.4708** Hz (**667.3584** cents)	**48.4466** Hz (**344.5578** cents)	**2.135771**
440 Hz	**113.4975** Hz (**397.2880** cents)	**72.1932** Hz (**263.0214** cents)	**1.572137**
880 Hz	**149.4869** Hz (**271.6201** cents)	**119.6863** Hz (**220.7663** cents)	**1.248989**
1760 Hz	**263.1640** Hz (**241.2454** cents)	**214.6726** Hz (**199.2457** cents)	**1.225885**

At 440 Hz, Zwicker's band is 3.972880 semitones and the equivalent rectangular bandwidth 2.630214.

> **Observation 3.2**
>
> **this is the core: the boundary is on the listener's side and moves with the definition.Both formulas are standard, and at 440 Hz they differ by 1.572137. The quantity on the side of the sound is fixed to 12 digits, while the boundary moves by half again with the choice of definition——"heard as two" is not written into the sound.**

Computation 3.3: **if beats down to 1 Hz can be heard, tuning can be brought within 3.930158 cents at 440 Hz and 6.604606 cents at 261.6256 Hz.**

4. The Separator

> **Observation 4.1**
>
> **the separator is whether the frequency difference exceeds the critical band.Exceed it and the two are heard apart; fall short and one tone wavers. The same two tones can change their answer by being moved to another register, where the bandwidth differs——with nothing changed on the side of the sound.**

Computation 4.2: **the errors of equal temperament shown in Paper 466 were turned into beat rates. Above 440 Hz the just fifth is 660.000000 Hz against the tempered 659.255114 Hz, a difference of 0.744886 Hz (1.955001 cents); the just major third is 550.000000 Hz against 554.365262 Hz, a difference of 4.365262 Hz (13.686286 cents).**

> **Observation 4.3**
>
> **beside Paper 466.Paper 466 stated that it looked only at ratios, and put the error of the tempered major third at 13.686286 cents——the same error appears above 440 Hz as 4.365262 beats per second. Cents are a ratio and beats are a difference: one discrepancy with two faces.**

> **Observation 4.4**
>
> **beside Paper 649.In Paper 649 the beat of two tidal constituents (14.7653 days) was a period in the signal, and separating them required a record as long——here too the beat is in the signal. What differs is that the separator is an ear rather than a computation, and so the boundary depends on a definition.**

> **Observation 4.5**
>
> **what is not said.Consonance, the merits of temperaments and individual differences are not treated, and no judgement is made between the two formulas——what is shown is that the beat is fixed to 12 digits on the side of the sound while the boundary for hearing two tones moves by 1.572137 on the side of the listener.**

5. Roll Call and Closing

1. **the identity agrees to 12 digits** (Computation 2.1): **the side of the sound.**
2. **peaks at 3 per second** (Observation 2.2): **envelope at 1.5 Hz.**
3. **1 Hz is 3.930158 cents** (Computation 2.3): **at 440 Hz.**
4. **15.667383 cents at 110 Hz** (Computation 2.3): **a ratio of 7.968379.**
5. **Zwicker 113.4975 Hz** (Computation 3.1): **the band at 440 Hz.**
6. **equivalent rectangular 72.1932 Hz** (Computation 3.1): **a ratio of 1.572137.**
7. **397.2880 cents** (Computation 3.1): **3.972880 semitones.**
8. **the fifth beats at 0.744886 Hz** (Computation 4.2): **1.955001 cents.**
9. **the third at 4.365262 Hz** (Computation 4.2): **13.686286 cents in Paper 466.**
10. **exceeding the critical band** (Observation 4.1): **the separator; Paper 649.**

Closing: **Whether two tones are heard as two, or as one tone wavering, is not written into the sound.The beat rate follows from an identity and agreed to 12 digits numerically.But the critical bandwidth that sets the boundary belongs to the listener, and at 440 Hz it is 113.4975 Hz by one standard formula and 72.1932 Hz by another, 1.572137 times apart——fixed to 12 digits on one side, moving by half again with a definition on the other.And the same 1 Hz beat is 15.667383 cents at 110 Hz and 1.966194 cents at 880 Hz——because a beat is a difference and an interval is a ratio.The separator is whether the frequency difference exceeds the critical band.Placed among the earlier papers——in the seat Paper 466 left open by not discussing hearing, its 13.686286 cents became 4.365262 beats per second; and where the beat of Paper 649 was a period in a signal, here the separator is an ear.To be honest ——two formulas were taken from the literature and no judgement was made between them.**

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