

The Qubit as a Hopf Fibration

Is Born’s “2” the Same Root as the Hopf Map’s “2”?

— A Disciplined Census of the Twos — [Paper 68]

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— *From thinking to syncing.* —

Abstract

The state space of a single qubit is exactly the Hopf fibration: normalised state vectors form $\mathcal{S}^3 \subset \mathbb{C}^2$, the physically irrelevant global phase is the fibre $U(1) = \mathcal{S}^1$, and the quotient $\mathcal{S}^3/U(1) = \mathcal{S}^2$ is the Bloch sphere; the projection $\mathcal{S}^3 \rightarrow \mathcal{S}^2$ is the Hopf map (Mosseri–Dandoloff 2001; Urbantke 2003). We observe that the Hopf (Bloch) map $(\alpha, \beta) \mapsto \langle \psi | \vec{\sigma} | \psi \rangle$ is *quadratic* in the amplitudes, and so is the Born rule $p = |\psi|^2$. **Proposition:** both are instances of the single sesquilinear pairing $\langle \psi | \mathbf{A} | \psi \rangle$ (with $\mathbf{A}$ Hermitian)—so *Born’s “2” and the Hopf map’s “2” are the same root*: the amplitude enters twice, once conjugated. This is the one place where two of the system’s “twos” genuinely coincide in origin. We then apply the discipline of Papers 48 and 58 to the remaining twos and give a **census**: the spatial-flux $n = 2$ (Papers 01, 53), the Euler characteristic $\chi(\mathcal{S}^2) = 2$ (Paper 67), the trace-reversal/ladder **2** (Paper 58), and the spinor/double-cover **2** (Berry phase = $\frac{1}{2}\Omega$; the **720°** periodicity) are shown to *coincide in value without sharing an origin*. The geometric (Berry) phase is identified as the holonomy of the Hopf fibre and ties to the solid angle of Paper 67, while remaining a distinct (spinor) root. The value of the census is precisely that it unifies only what shares a root and separates the rest.

Keywords: Hopf fibration, qubit, Bloch sphere, Born rule, Berry phase, sesquilinear form, distinct roots, structural interpretation.

1. Introduction

Paper 67 reached Hopf through the Euler-characteristic door, anchoring $4\pi = 2\pi\chi(\mathcal{S}^2)$ topologically. This paper takes the second sound door recommended after the mass–instanton “**16π²**” bridge fell (Paper 15 note): the **spin/phase** door. There, Hopf appears not as an analogy but as the literal geometry of a two-level quantum system.

The state space of a qubit *is* the Hopf fibration. Once this is on the table, a question in the spirit of Paper 58 becomes unavoidable: the Yamagishi system is full of factors of two—the spatial $n = 2$ (Paper 01), Born’s $|\psi|^2$ (Paper 53), the trace-

reversal **2** (Paper 58), the Euler characteristic $\chi(S^2) = 2$ (Paper 67)—and the Hopf map adds another. *Which of these share a root, and which merely coincide in value?* Answering this is the work of the paper.

What this paper does not do. It does not derive the Born rule (quantum mechanics is assumed) and claims no new physics. It reports one genuine unification and, more importantly, resists the rest.

2. The Qubit Is the Hopf Fibration

A pure qubit state is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1, \quad (1)$$

so $(\alpha, \beta) \in S^3 \subset \mathbb{C}^2$. The global phase $|\psi\rangle \sim e^{i\gamma}|\psi\rangle$ is physically irrelevant; quotienting by this $U(1) = S^1$ gives the space of physical states,

$$S^3/U(1) = \mathbb{CP}^1 = S^2 \quad (\text{the Bloch sphere}), \quad (2)$$

and the projection $\pi : S^3 \rightarrow S^2$ is the **Hopf map**. Explicitly, with $\vec{\sigma}$ the Pauli matrices,

$$\pi(\alpha, \beta) = (2\operatorname{Re} \bar{\alpha}\beta, 2\operatorname{Im} \bar{\alpha}\beta, |\alpha|^2 - |\beta|^2) = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle) = \vec{n}, \quad (3)$$

the Bloch vector, $|\vec{n}| = 1$ (Hopf, 1931; Mosseri–Dandoloff, 2001; Urbantke, 2003).

3. Born and Hopf Share One Root

Proposition 3.1 (Born’s 2 = the Hopf map’s 2)

Both the Born probabilities and the Bloch-vector (Hopf-map) components are values of the single sesquilinear pairing $\langle\psi|A|\psi\rangle$ with A Hermitian:

$$\underbrace{p_0 = |\alpha|^2 = \langle\psi|\Pi_0|\psi\rangle, \quad p_1 = |\beta|^2 = \langle\psi|\Pi_1|\psi\rangle}_{\text{Born}}, \quad \underbrace{n_i = \langle\psi|\sigma_i|\psi\rangle}_{\text{Hopf/Bloch}},$$

where Π_0, Π_1 are the projectors onto $|0\rangle, |1\rangle$. In each case the amplitude ψ enters exactly twice, once conjugated. Hence the exponent **2** of Born’s rule and the quadratic degree of the Hopf map are *the same root*: the sesquilinearity of quantum theory. Moreover $p_0 = (1 + n_z)/2$, so the Born probabilities are affine images of the Hopf map’s third component.

Proof.

$|\alpha|^2 = \langle \psi | \Pi_0 | \psi \rangle$ and $|\beta|^2 = \langle \psi | \Pi_1 | \psi \rangle$ by definition of the projectors; $n_i = \langle \psi | \sigma_i | \psi \rangle$ by (3). All are of the form $\langle \psi | A | \psi \rangle$, quadratic (sesquilinear) in ψ . The identity $p_0 = (1 + n_z)/2$ follows from $n_z = |\alpha|^2 - |\beta|^2$ and $|\alpha|^2 + |\beta|^2 = 1$. ■

This is the single genuine coincidence of *origin* among the system’s twos: on the qubit, “the state is squared to give a physical quantity” is one structure, realised both as Born probability and as Bloch coordinate.

4. A Census of the Twos (Paper 58 Discipline)

Having found one shared root, the discipline of Papers 48 and 58 forbids assuming the others follow. We tabulate.

The “2”	Where it appears	Its root
Sesquilinear 2	Born $ \psi ^2$; Hopf/Bloch $\langle \psi \sigma \psi \rangle$	quantum pairing $\langle \psi A \psi \rangle$
Spatial-flux $n = 2$	Paper 01; r^{-2} dilution	3D geometry (solid angle / flux)
Topological $\chi(S^2) = 2$	Paper 67	topology (Euler characteristic)
Trace-reversal / ladder 2	Paper 58	GR trace / dimensional ladder
Spinor / double-cover 2	Berry = $\frac{1}{2}\Omega$; 720°	$SU(2) \rightarrow SO(3)$ (spin- $\frac{1}{2}$)

Proposition 4.1 (Only Born and Hopf share a root)

Of the five twos above, exactly one pair—Born and the Hopf/Bloch map—shares an origin (Proposition 3.1). The remaining twos coincide in value but not in origin:

- The spatial $n = 2$ is a statement about 3D falloff; it meets the sesquilinear 2 only by *composition*: for an outgoing spherical wave $\psi \sim e^{ikr}/r$, squaring ($|\psi|^2$, the sesquilinear 2) applied to $\psi \sim 1/r$ produces r^{-2} (the spatial 2), as noted in Paper 53. Composition is not identity.
- The Euler characteristic $\chi(S^2) = 2$ is topological (Paper 67); it does not involve amplitudes.
- The trace-reversal 2 is a feature of the Einstein tensor / dimensional ladder (Paper 58), crossing the ladder factor only at $D = 4$.
- The spinor 2 (below) is the covering degree $SU(2) \rightarrow SO(3)$.

No further identification is claimed.

The point of the census is not to unify the twos but to *stop* unifying them. One coincidence of origin (Born=Hopf) is real; the others are resonances in value.

5. The Berry Phase Is the Hopf Fibre's Holonomy

The fibre $U(1)$ of the Hopf map carries a natural connection whose holonomy around a loop C on the Bloch sphere is the geometric (Berry) phase (Berry, 1984; Simon, 1983):

$$\gamma(C) = -\frac{1}{2} \Omega(C), \quad (4)$$

half the solid angle $\Omega(C)$ enclosed by C . The factor $\frac{1}{2}$ is the spinor/double-cover root: a spin- $\frac{1}{2}$ state requires a 4π (720°) rotation to return to itself. This ties to Paper 67—the solid angle Ω is the same 4π -normalised quantity whose total value is $2\pi\chi(S^2)$ —but the $\frac{1}{2}$ here is a *distinct* (spinor) root, not the sesquilinear or topological one. The Berry phase is measurable, so the Hopf fibre is not a formal device but an observable structure.

6. Scope of the Claims

What is established:

1. The qubit state space is the Hopf fibration $S^3 \rightarrow S^2$ (Section 2).
2. Born's exponent and the Hopf/Bloch map share one root, quantum sesquilinearity $\langle\psi|A|\psi\rangle$ (Proposition 3.1).

3. Among the system's five twos, only that pair shares an origin; the rest coincide in value only (Proposition 4.1). The Berry phase is the Hopf fibre's holonomy, $\gamma = -\frac{1}{2}\Omega$ (a distinct spinor root).

What is not established / not claimed:

- A derivation of the Born rule. Quantum mechanics (states, projectors, the pairing $\langle\psi|A|\psi\rangle$) is assumed.
- Any unification of the five twos beyond the single Born=Hopf coincidence. The census is designed to prevent over-unification.
- Any use of the mass–instanton “ $16\pi^2$ ” bridge, which remains distinct (Paper 15). This paper uses only the spin/phase door.

Open problem. Is there a reason the two-level system's geometry is *exactly* the Hopf fibration—i.e., does $SU(2)$ (spin- $\frac{1}{2}$) force the $S^3 \rightarrow S^2$ structure uniquely, or is it one realisation among several? Relatedly, for a two-qubit system the relevant object is the $S^7 \rightarrow S^4$ (quaternionic) Hopf fibration and entanglement (Mosseri–Dandoloff); whether the census of twos extends there is left open.

7. Conclusion

The qubit is the Hopf fibration: state vectors on S^3 , global phase the fibre $U(1)$, physical states the Bloch sphere S^2 . On it, Born's “2” and the Hopf map's “2” are one and the same root—the sesquilinear pairing $\langle\psi|A|\psi\rangle$ —the single genuine coincidence of origin among the system's factors of two. Applying the discipline of Papers 48 and 58, the remaining twos (spatial $n = 2$, topological $\chi = 2$, trace-reversal 2, spinor 2) coincide in value but not in origin. The Berry phase realises the Hopf fibre's holonomy as a measurable half-solid-angle, linking to Paper 67 while staying a distinct spinor root. Reached through the spin/phase door, this honours Hopf without touching the distinct mass–instanton $16\pi^2$, and its value lies in separating one true root from four resonances.

Revision Record

Second edition (September 2026): no claim, theorem or check has been altered from the first edition. German, French and Italian versions have been added. The references that cite earlier papers of this corpus now give **the actual titles of the cited papers in the current format** (the paper number and DOI after the title); the first edition had shortened the titles. The paper number has been moved onto the title line, and the DOI has been removed from the date line. Papers 15 and 48, cited in the text, have been added to the references. **Nothing of substance has changed, and nothing has been withdrawn.**

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